Topological Properties of Cold Magnetized Plasma

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1 Introduction

The focus of this paper is on cold plasma, which is characterized by a low enough temperature that the motion of ions other than electrons is neglected-essentially an electron gas. This approximation has various applications in low energy plasma physics [11][4][8] and photonics [3] [7] [14]. The wave dynamics for cold plasma and plasmas in general has been well studied [15], but recent advances in topological physics suggest that topological arguments may be able to predict novel behaviors in plasmas. For a review of the topological physics in general and as applied to plasma physics see [6] [13] [9].

Recently, one topologically protected edge state at the edge between two regions of topologically distinct electron densities, termed the Topological Langmuir Cyclotron Wave (TLCW) has been studied. Parker et. al. first predicted the mode using numerical methods in [11], and Fu and Qin provided a detailed analysis of this mode in [12] [1] and characterized the topological phases of the system in [2].

The preferred topological invariants used in this system and other hydrodynamic like systems are Chern numbers. The main goal of this paper is to expand on the results of [2] [12] [11] by analytically calculating all the Chern numbers of the system, which were previously only calculated numerically. It is apparent from the fact that some Chern numbers are not integer-valued that the Bulk-Edge Correspondence (BEC) does not exist for this system without some modification. Although we do not attempt to prove the existence or non-existence of the BEC here, some intuition regarding it's validity is analyzed in Section 5 by comparing with some numerical results, particularly in the case where some regularization factors are added in.

Some analytic calculations of Chern numbers have been done for particular parameter values ($k_z = 0$) [5] which are of particular interest in photonics. In this case we shall see that the system as we have defined it actually breaks into two de-coupled systems and fundamentally alters the topological structure.

We will begin by deriving the Hamiltonian of the system and analyzing its symmetry properties. Then, all the eigenvectors needed to calculate Chern numbers are derived. Although analytical expressions are not available for eigenvectors in the general case, we will find that only certain limits of eigenvectors are needed to calculate the Chern numbers of this system [14]. Applying these results allows us to calculate some Chern numbers directly and infer all others from symmetries of the system. Finally, the $k_z = 0$ case is analyzed independently and compared to the general case, and some numerical results are presented to develop some intuition on whether a BEC can exist in general.

1.1 Derivation of Equations

The following derivation largely mirrors the derivation in [15]. We are interested here in waves in a plasma biased by a constant incident magnetic field $\mathbf{B}_0 = \hat{z}B_0$. We start with Maxwell's equations and Lorentz force equation for a cloud of electrons with density n_e, velocity v, and charge q_e. In order to linearize the problem, only the incident field is considered in Lorentz's equation, which is merely to assume that the magnitude of any waves present in the plasma are small compared to the incident field.

$$m_e \frac{\partial v}{\partial t} = q_e(E + v \times B_0)$$

$$c^2 \nabla \times B = \frac{1}{\epsilon_0} J + \frac{\partial E}{\partial t} = \frac{n_e q_e v}{\epsilon_0} + \frac{\partial E}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Note that we treat the electron density as a constant, or slowly varying with respect to frequency, mean density about which the electrons essentially "vibrate". SI units are used above, but we will re-normalize the system so that v, E, and k all have units of electric field $\frac{m \cdot kg}{s^2 \cdot C} = N/C$. Now we will make the substitutions:

$$v_n = v \frac{m_e \omega_p}{q_e}$$

$$B_n = cB$$

We define the electron plasma frequency and electron gyro-frequency respectively as:

$$\omega_p = \sqrt{\frac{n_e q_e^2}{m_e \epsilon_0}}$$

$$\Omega = -\frac{B_0 q_e}{m_e}$$

Note that since $q_e < 0$ we have that Ω is the same sign as B_0. This gives the equations:

$$\partial_t v_n = \omega_p E + \Omega \hat{z} \times v_n$$

$$\partial_t E = c \nabla \times B_n - \omega_p v_n$$

$$\partial_t B_n = -c \nabla \times E$$

For the majority of the paper we will be concerned with obtaining and analyzing eigenvalues of the system, so we take the Fourier transform in space and time to obtain:

$$i\omega\tilde{v_n} = \omega_p\tilde{E} + \Omega\hat{z}\times\tilde{v}_n$$

$$i\omega\tilde{E} = -ick\times\tilde{B}_n - \omega_p\tilde{v}_n$$

$$i\omega\tilde{B} = ick\times\tilde{E}$$

Finally normalizing $k_n = ck$ we get:

$$\omega\tilde{v_n} = -i\omega_p\tilde{E} - i\Omega\hat{z}\times\tilde{v}_n$$

$$\omega\tilde{E} = -k_n\times\tilde{B}_n + i\omega_p\tilde{v}_n$$

$$\omega\tilde{B} = k_n\times\tilde{E}$$

For the rest of this paper we will drop the subscripts and $\tilde{x}$ notation and simply regard v, E, B as the normalized Fourier transform of themselves as defined above. This gives the eigenvalue problem:

$$H\begin{bmatrix} v \\ E \\ B \end{bmatrix} = \omega\begin{bmatrix} v \\ E \\ B \end{bmatrix}$$

For the 9x9 Hamiltonian:

$$H = \begin{bmatrix} -i\Omega\hat{z}\times & -i\omega_p I & 0 \\ i\omega_p I & 0 & -k\times \\ 0 & k\times & 0 \end{bmatrix} \tag{1}$$

Here and throughout we use the shorthand for the cross product matrix:

$$u\times = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

for $u \in \mathbb{C}^3$. Because electron motion in the $\hat{z}$ direction is unaffected by the incident magnetic field, k_z can be treated as a parameter of the system along with Ω- proportional to B_0- and ω_p- proportional to $\sqrt{n_e}$. For a detailed discussion of the validity of this assumption and the cold-plasma approximation in general see [15].

1.2 Susceptibility Tensor

In many cases it will be useful to eliminate v from the system using the susceptibility tensor χ such that $v = \chi E$. Writing out the v row of H gives:

$$i\Omega\hat{z}\times v - i\omega_p E = \omega v$$

Now make substitute the circularly left and right polarized vectors:

$$v_{\pm} = e_{\pm} = \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

Then we get:

$$-\Omega \hat{z} \times v_{\pm} + i\omega v_{\pm} = \omega_p E_{\pm} \Rightarrow$$

$$\Omega \begin{pmatrix} \pm i \\ -1 \\ 0 \end{pmatrix} + i\omega \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} = \begin{pmatrix} i(\omega \pm \Omega) \\ -\Omega \mp \omega \\ 0 \end{pmatrix} = \begin{bmatrix} i\omega & \Omega & 0 \\ -\Omega & i\omega & 0 \\ 0 & 0 & 0 \end{bmatrix} v_{\pm} = \omega_p E_{\pm}$$

Similarly if we assume that $v = \hat{z}$ then

$$-\Omega \hat{z} \times v + i\omega v = i\omega v = \omega_p E \Rightarrow$$

$$E_z = i\frac{\omega}{\omega_p} v_z$$

Therefore we have that:

$$\frac{1}{\omega_p} \begin{bmatrix} i\omega & \Omega & 0 \\ -\Omega & i\omega & 0 \\ 0 & 0 & i\omega \end{bmatrix} v = E$$

For finite ω_p, Ω, and ω this matrix is invertible as long as $\omega \neq \Omega$. Therefore assuming that $\omega \neq \Omega$ we get:

$$\chi = \left(\frac{1}{\omega_p} \begin{bmatrix} i\omega & \Omega & 0 \\ -\Omega & i\omega & 0 \\ 0 & 0 & i\omega \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{i\omega_p \omega}{\Omega^2 - \omega^2} & -\frac{\omega_p \Omega}{\Omega^2 - \omega^2} & 0 \\ \frac{\omega_p \Omega}{\Omega^2 - \omega^2} & \frac{i\omega_p \omega}{\Omega^2 - \omega^2} & 0 \\ 0 & 0 & -i\frac{\omega_p}{\omega} \end{bmatrix}$$

If we substitute this into (1) the first line is eliminated (we used it to derive χ) and the Hamiltonian becomes:

$$H = \begin{bmatrix} i\omega_p \chi & -k\times \\ k\times & 0 \end{bmatrix} \tag{2}$$

$$H \begin{bmatrix} E \\ B \end{bmatrix} = \omega \begin{bmatrix} E \\ B \end{bmatrix}$$

2 Eigenvalue Symmetry

2.1 $\pm\omega$ Symmetry

Below we prove that the eigenvalues of $\hat{H}$ have symmetry in $\pm n$ for $n = \{-4, -3, ..., 3, 4\}$. In particular if:

$$Hx = \omega_n x \tag{3}$$

$$x = \begin{bmatrix} v \\ E \\ B \end{bmatrix}$$

then

$$H \begin{bmatrix} v^* \\ E^* \\ -B^* \end{bmatrix} = -\omega_n \begin{bmatrix} v^* \\ E^* \\ -B^* \end{bmatrix}$$

First notice that the cross product matrix as defined above is real anti-symmetric:

$$\mathbf{k}\times = -(\mathbf{k}\times)^* = -(\mathbf{k}\times)^T$$

Assume that (3) holds. By straightforward computation we have:

$$\hat{H} \begin{bmatrix} v^* \\ E^* \\ -B^* \end{bmatrix} = \begin{bmatrix} -i\Omega\hat{z} \times v^* - i\omega_p E^* \\ i\omega_p v^* + \mathbf{k} \times B^* \\ \mathbf{k} \times E^* \end{bmatrix}$$

By the properties of the cross product from above we have that:

$$\hat{z} \times v^* = ((v^*)^\dagger (\hat{z}\times)^\dagger)^\dagger = (v^T (\hat{z}\times)^T)^\dagger = (\hat{z} \times v)^*$$

and likewise:

$$\mathbf{k} \times B^* = (\mathbf{k} \times B)^*$$

$$\mathbf{k} \times E^* = (\mathbf{k} \times E)^*$$

From (3) we get that:

$$\mathbf{k} \times E = \omega_n B \Rightarrow$$

$$\mathbf{k} \times E^* = (\mathbf{k} \times E)^* = (\omega_n B)^* = \omega_n B^*$$

where we know that ω_n is real since H is Hermitian. Likewise from (2) we get:

$$\omega_n E = i\omega_p v - \mathbf{k} \times B = i(\omega_p \mathrm{Re}(v) - \mathrm{Im}(\mathbf{k} \times B)) - (\omega_p \mathrm{Im}(v) + \mathrm{Re}(\mathbf{k} \times B))$$

Comparing with $(v^*, E^*, -B^*)$:

$$i\omega_p v^* + \mathbf{k}\times B^* = i(\omega_p \mathrm{Re}(v) - \mathrm{Im}(\mathbf{k}\times B)) + (\omega_p \mathrm{Im}(v) + \mathrm{Re}(\mathbf{k}\times B)) = -(i\omega_p v - \mathbf{k}\times B)^* = -\omega_n E^*$$

Similarly for the velocity component:

$$\omega_n v = -i\Omega\hat{z} \times v - i\omega_p E = -i(\Omega \mathrm{Re}(\hat{z} \times v) + \omega_p \mathrm{Re}(E)) + (\Omega \mathrm{Im}(\hat{z} \times v) + \omega_p \mathrm{Im}(E))$$

$$-i\Omega\hat{z}\times v^* - i\omega_p E^* = -i(\Omega \mathrm{Re}(\hat{z}\times v^*) + \omega_p \mathrm{Re}(E)) - (\Omega \mathrm{Im}(\hat{z}\times v) + \omega_p \mathrm{Im}(E)) = -(-i\Omega\hat{z}\times v - i\omega_p E)^* = -\omega_n v^*$$

$$\hat{H} \begin{bmatrix} v^* \\ E^* \\ -B^* \end{bmatrix} = -\omega_n \begin{bmatrix} v^* \\ E^* \\ -B^* \end{bmatrix} \tag{4}$$

or in other words the eigenvalues of $\hat{H}$ are symmetric around 0 and their associated eigenvectors are obtained by conjugation and reflection across the B-plane.

2.2 $\pm\Omega$ Symmetry

From (1) we can see that:

$$\begin{bmatrix} -I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} H(\Omega) \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix} = -\Gamma_\Omega H(\Omega)\Gamma_\Omega = H(-\Omega)$$

Γ_Ω is clearly orthogonal so:

$$\Gamma_\Omega H(\Omega) = -H(-\Omega)\Gamma_\Omega$$

Assuming (3) holds this gives:

$$\Gamma_\Omega H(\Omega)x = \Gamma_\Omega \omega_n x = -H(-\Omega)\Gamma_\Omega x \Rightarrow$$

$$H(-\Omega)(\Gamma_\Omega x) = -\omega_n(\Gamma_\Omega x)$$

Therefore if (3) holds then $-\omega_n$ is an eigenvalue of $H(-\Omega)$ with eigenvector:

$$x = \begin{bmatrix} v \\ -E \\ B \end{bmatrix}$$

Combining this fact with (4) we get that ω_n is also an eigenvector of $H(-\Omega)$ with eigenvector:

$$\begin{bmatrix} v^* \\ -E^* \\ -B^* \end{bmatrix}$$

2.3 $\pm\mathbf{k}$ Symmetry

Now define the projection:

$$\Gamma_k = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}$$

Then we have:

$$H(-\mathbf{k}) = \Gamma_k H(\mathbf{k})\Gamma_k$$

Similarly to the previous subsection then:

$$\omega_n(\Gamma_k x) = H(-\mathbf{k})(\Gamma_k x)$$

so we have that ω_n is an eigenvalue of $H(-\mathbf{k})$ with eigenvector:

$$\begin{bmatrix} v \\ E \\ -B \end{bmatrix}$$

and from (4) we know that $-\omega_n$ is an eigenvalue of $H(-\mathbf{k})$ with eigenvector:

$$\begin{bmatrix} v^* \\ E^* \\ B^* \end{bmatrix}$$

2.4 k_x/k_y Plane Rotational Symmetry

Consider rotating the $\mathbf{k}$ vector in the x/y plane. Suppose that after $\mathbf{k}$ is rotated in the x/y plane by and angle θ, the new Hamiltonian is H_θ. Using the usual 2-d rotation matrix we can see that if we rotate k by an angle θ in the x/y plane we get:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} k_x\cos\theta - k_y\sin\theta \\ k_x\sin\theta + k_y\cos\theta \\ k_z \end{bmatrix}$$

Denote the new rotated components $k_{x2} = k_x\cos\theta - k_y\sin\theta$ and $k_{y2} = k_x\sin\theta + k_y\cos\theta$. Also denote the rotation matrix:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One can easily show that R is orthonormal. Remembering the definition of $\mathbf{k}\times$:

$$\mathbf{k}\times = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

we can see that we want to rotate the vector $\begin{bmatrix} k_y & -k_x & 0 \end{bmatrix}^T$ in the third column and the same corresponding column vector in the third row.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} k_x \\ k_y \end{bmatrix} = S \begin{bmatrix} k_x \\ k_y \end{bmatrix} = \begin{bmatrix} k_y \\ -k_x \end{bmatrix}$$

Noticing that $S^{-1} = -S$ we can see that x/y plane rotation by θ can be accomplished on the $(k_y, -k_x)^T$ vector by the transformation:

$$-SRS \begin{bmatrix} k_y \\ k_x \end{bmatrix} = \begin{bmatrix} k_{y2} \\ -k_{x2} \end{bmatrix}$$

Now due to the fact that R is anti-symmetric, it commutes with S, so $-SRS = -SSR = R$. Rotating the last row would be equivalent to the operation:

$$\left(R \begin{bmatrix} k_y \\ -k_x \end{bmatrix} \right)^T = \begin{bmatrix} k_y & -k_x \end{bmatrix} R^T$$

Noticing that the upper left portion of $\mathbf{k}\times$ is just a factor of S we can see that:

$$R[\mathbf{k}\times]R^T = \begin{bmatrix} 0 & -k_z & k_{y2} \\ k_z & 0 & -k_{x2} \\ -k_{y2} & k_{x2} & 0 \end{bmatrix}$$

which can be verified by straightforward calculation. Therefore denoting:

$$\mathbf{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}$$

and noticing that $\hat{z}\times$ is also a factor of S and using the fact that R is orthonormal we get:

$$H_\theta = \mathbf{R}H\mathbf{R}^T$$

This derivation was shown to highlight the symmetries of the problem but straightforward computation also verifies this fact. Assume that x is and eigenvector of H with eigenvalue ω, or $Hx = \omega$. Multiplying the above equation on the right by $\mathbf{R}$ gives:

$$H_\theta \mathbf{R} = \mathbf{R}H \Rightarrow$$

$$H_\theta(\mathbf{R}x) = \mathbf{R}Hx = \omega(\mathbf{R}x)$$

Therefore ω is also an eigenvalue of H_θ with eigenvector $\mathbf{R}x$, which simply rotates the x and y components of v, E, B respectively by θ.

3 Eigenvector Calculations

As shown above, the eigenvalues have well-defined $\pm$ symmetry, so we will only show derivations for the positive and zero-valued eigenvalues. First note that there is a zero eigenvalue for any parameter choice which is $\Psi = (0, 0, k)$. Plugging this Ansatz into (1) we get:

$$k \times E = k \times 0 = 0$$

$$i\omega_p v - k \times B = -k \times k = 0$$

$$-i\Omega\hat{z} \times v - i\omega_p E = 0$$

regardless of the values of $k_z, k_\perp, \omega_p, \Omega$. Therefore we have the eigenvalue/eigenvector pair:

$$\omega_0 = 0$$

$$\Psi_0 = (0, 0, k)$$

Unfortunately this is the only readily available eigenvalue/eigenvector pair which applies to all parameter values. Below we consider some instructive and especially useful cases where eigenvalues can be analytically derived.

3.1 Simplest Case: $\Omega = 0$

The non-negative eigenvalues and eigenvectors are shown below. Due to $\pm\omega$ symmetry as proved above all bands are symmetric around $\omega = 0$ so it is sufficient to consider only non-negative bands. Detailed calculations are shown in

Appendix A.

$$
\begin{aligned}
\omega_0 &= 0 \\
\Psi_0 &= (-ik \times B, 0, \omega_p B)(3) \\
\omega_1 &= \omega_p \\
\Psi_1 &= (k, ik, 0) \\
\omega_2 &= \sqrt{k^2 + \omega_p^2} \\
\Psi_2 &= (\omega_p v, i\omega_2 v, ik \times v)(2)
\end{aligned}
\tag{5}
$$

Here B can be chosen to be any orthonormal basis of $\mathbb{R}^3$ to produce three degenerate eigenvalues at $\omega_0 = 0$ and v can be chosen to be any orthonormal basis perpendicular to k to produce two degenerate eigenvectors for $\omega_2 = \sqrt{k^2 + \omega_p^2}$. These degenerate eigenvalues show that a topologically protected edge state is possible around the boundary $\Omega = 0$ when varying Ω.

3.2 Case: $k_\perp = 0$

The $k_\perp = 0$ case is important for two reasons. First, band crossings only happen when $k_\perp = 0$, $\Omega = 0$, or $k_z = 0$, so these parameter values are important for considering topologically protected edge states. Second, as we will see later, the $k_\perp = 0$ eigenvectors are essential for calculating Chern numbers. Calculations are presented below along with analysis of band crossings that will be especially useful in calculting Chern numbers.

Setting $k_\perp = (k_x, k_y) = 0$ gives us the eigenvalue equation from (1):

$$
\begin{aligned}
-i\Omega\hat{z} \times v - i\omega_p E &= \omega v \\
i\omega_p v - k_z\hat{z} \times B &= \omega E \\
k_z\hat{z} \times E &= \omega B
\end{aligned}
\tag{6}
$$

We will consider the non-trivial case $k_z \neq 0$. Consider the Ansatz for a plasma oscillation, or $\omega_1 = \omega_p$. From the first line of (6) we get:

$$-i\Omega\hat{z} \times v - i\omega_p E = \omega_p v$$

If we consider the case that v is either real or purely imaginary it's apparent from this equation that $i\Omega\hat{z} \times v \perp \omega_p v$. Therefore a non-trivial solution is $E = \hat{z}$ and $v = -i\hat{z}$ It follows from the rest of (6) that:

$$\omega_p B = k_z\hat{z} \times E = k_z\hat{z} \times \hat{z} = 0 \Rightarrow B = 0$$

$$i\omega_p v - k_z\hat{z} \times B = \omega_p\hat{z} = \omega_p E$$

Therefore we have the plasma eigenmode:

$$\omega_1 = \omega_p$$

$$\Psi_1 = (\frac{\hat{z}}{\sqrt{2}}, \frac{i\hat{z}}{\sqrt{2}}, 0)$$

For the remaining 3 positive eigenvalues we will utilize the susceptibility tensor calculated in Section 1.1 $v = \chi E$. As we showed that the vectors right and left circularly polarized vectors in the x/y plane $E_\pm = v_\pm = (1, \pm i, 0)$ are eigenvectors of χ these may be good Ansatz's for eigenvectors of the whole system.

With some rearranging of (2) we get:

$$k \times E = \omega B \Rightarrow k \times B = \frac{k}{\omega} \times (k \times E)$$

$$\omega(I - i\frac{\omega_p}{\omega}\chi)E + k \times B = 0 \Rightarrow$$

$$\frac{k}{\omega} \times (\frac{k}{\omega} \times E) + (I - i\frac{\omega_p}{\omega}\chi)E = 0$$

Setting $I - i(\omega_p/\omega)\chi = \epsilon$ and $n = k/\omega$ this gives what is commonly known as the homogeneous plasma wave equation:

$$n \times (n \times E) + \epsilon E = 0 \tag{7}$$

The dielectric tensor ϵ is commonly written as:

$$\epsilon = \begin{bmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

where one can check easily that:

$$R = 1 - \frac{\omega_p^2}{\omega(\omega + \Omega)}$$

$$L = 1 - \frac{\omega_p^2}{\omega(\omega - \Omega)}$$

$$S = \frac{1}{2}(R + L)$$

$$D = \frac{1}{2}(R - L)$$

$$P = 1 - \frac{\omega_p^2}{\omega^2}$$

Assuming WLOG that $k_y = n_y = 0$ and denoting θ as the angle between n and $\hat{z}$ we can represent the operator $[n \times (n \times]$ in matrix form and write the plasma wave equation as:

$$\begin{bmatrix} S - n^2\cos^2\theta & iD & n^2\cos\theta\sin\theta \\ -iD & S - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P - n^2\sin^2\theta \end{bmatrix} E = 0 \tag{8}$$

This is in fact the dispersion relation for the whole system, however it is not solvable in closed form for arbitrary θ. For the current case $k = k_z\hat{z} \Rightarrow \theta = 0$ and the dispersion relation gives:

$$\begin{bmatrix} S - n^2 & iD & 0 \\ -iD & S - n^2 & 0 \\ 0 & 0 & P \end{bmatrix} E = 0$$

Note that the $P = 1 - \frac{\omega_p^2}{\omega^2} = 0$, $E = \hat{z}$ case has already been addressed. Therefore, as mentioned above we will guess that the remaining eigenmodes have E component polarized in the x/y plane. Assuming that $E_z = 0$ and taking the determinant of the upper left matrix to determine when it is singular gives the equation:

$$(S - n^2)^2 - D^2 = 0$$

Plugging in the definition of S and D gives the dispersion relation:

$$\frac{1}{4}(R + L)^2 + (n^2)^2 - n^2(L + R) - \frac{1}{4}(L - R)^2 = LR + n^2(n^2 - L - R) = 0$$

We can see that solutions to this equation are:

$$n^2 = R \Rightarrow$$

$$\left(\frac{k}{\omega}\right)^2 = 1 - \frac{\omega_p^2}{\omega(\omega + \Omega)} \Rightarrow$$

$$k^2 = \omega^2 - \frac{\omega_p^2}{1 + \frac{\Omega}{\omega}} \tag{9}$$

which is usually called the R-wave (right circularly polarized), and

$$n^2 = L \Rightarrow$$

$$\left(\frac{k}{\omega}\right)^2 = 1 - \frac{\omega_p^2}{\omega(\omega - \Omega)} \Rightarrow$$

$$k^2 = \omega^2 - \frac{\omega_p^2}{1 - \frac{\Omega}{\omega}} \tag{10}$$

which is usually called the L-wave (left circularly polarized). With some elementary manipulation we can see that the eigenvalues are roots of the cubic polynomials:

$$\omega_R^3 + \Omega\omega_R^2 - (k^2 + \omega_p^2)\omega_R - k^2\Omega = 0 \tag{11}$$

for the R-wave and:

$$\omega_L^3 - \Omega\omega_L^2 - (k^2 + \omega_p^2)\omega_L + k^2\Omega = 0 \tag{12}$$

for the L wave. Assuming that all the roots are real these account for the remaining six eigenvalues. Explicit formulas are possible as well, though they are complicated and not very instructive.

To illustrate some of the behavior of these eigenvalues first take the $k_z = 0$ case. Immediately we see the L and R modes share the eigenvalue $\omega = 0$ and the remaining values are the roots of the quadratic equations:

$$\omega^2 + \Omega\omega - \omega_p^2 = 0 \Rightarrow$$

$$\frac{\omega_R^0}{\omega_p} = \frac{-\sigma \pm \sqrt{\sigma^2 + 4}}{2} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + 1} - \frac{\sigma}{2}$$

for the R-waves and

$$\omega^2 - \Omega\omega - \omega_p^2 = 0 \Rightarrow$$

$$\frac{\omega_L^0}{\omega_p} = \frac{\sigma \pm \sqrt{\sigma^2 + 4}}{2} = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + 1}$$

for the L-waves. If we analyze (9) and (10) as $k \to \infty$ we can see that for positive ω the L-wave either has $\omega \to \infty$ or $\omega \to \Omega^-$ and the R-wave only supports $\omega \to \infty$. Therefore it's apparent from these equations that there are 2 positive L-waves. One which ω_L starts at 0 when $k = 0$ and goes to Ω as $k \to \infty$. This lower branch usually called the cyclotron mode [2] as it converges to the cyclotron frequency Ω. And one which starts at ω_L^0 and goes to ∞ as $k \to \infty$. There is one positive R-wave.

Analyzing the positive R-wave, since we are assuming that $\sigma > 0$ we have $(\sigma/2)^2 + 1 < (\sigma/2 + 1)^2 \Rightarrow \omega_R^0 < \omega_p$. From (9) we can also see that, assuming $\omega > 0$ we must have $\omega \to \infty$ for $k \to \infty$. Therefore we know that the positive R-wave crosses the Langmuir band. Plugging in $\omega = \omega_p = \omega_+$ into (9) gives:

$$k_z^2 = \omega_+^2 - \frac{\omega_+^2}{1 + \frac{\Omega}{\omega_+}} \Rightarrow \Omega\omega_+^2 - k_z^2\omega_+ - \Omega k_z^2 = 0 \Rightarrow$$

$$\omega_+ = \frac{k_z^2 + \sqrt{k_z^4 + 4\Omega^2 k_z^2}}{2\Omega}$$

$$\frac{\omega_+}{\Omega} = \frac{1}{2}\left(\sqrt{\left(\frac{k_z}{\Omega}\right)^4 + 4\left(\frac{k_z}{\Omega}\right)^2} + \left(\frac{k_z}{\Omega}\right)^2\right)$$

which characterizes the band crossing between the positive R-band and the Langmuir band.

Similarly if $\omega_p < \Omega$ ($\sigma > 1$) we can see that the cyclotron band (lower L-wave) crosses the Langmuir band (when $k_z = 0$, $\omega_L^- = 0$ and as $k_z \to \infty$, $\omega_L^- \to \Omega$). Denoting this crossing $\omega = \omega_p = \omega_-$ we get:

$$k_z^2 = \omega_-^2 - \frac{\omega_-^2}{1 - \frac{\Omega}{\omega_-}} \Rightarrow \Omega\omega_-^2 + k_z^2\omega_- - \Omega k_z^2 = 0 \Rightarrow$$

$$\omega_- = \frac{\sqrt{k_z^4 + 4\Omega^2 k_z^2} - k_z^2}{2\Omega} \Rightarrow$$

$$\frac{\omega_-}{\Omega} = \frac{1}{2}\left(\sqrt{\left(\frac{k_z}{\Omega}\right)^4 + 4\left(\frac{k_z}{\Omega}\right)^2} - \left(\frac{k_z}{\Omega}\right)^2\right)$$

By analyzing (9) and (10) we can see that as $\omega_p \to 0$ we must have $\omega_{L,R} \to k_z$. Therefore since ω_R and ω_L only cross ω_p at one point we must have that when $\omega_p < \omega_+$, $\omega_R > \omega_p$, when $\omega_p < \omega_-$, $\omega_L > \omega_p$, and similarly when $\omega_p > \omega_\pm$ $\omega_{R,L} < \omega_p$ respectively. This ordering is especially important when calculating Chern numbers below.

Numerical illustration of band crossings and eigenvalue behavior is shown in Figure 1. The ω_- crossing gives rise to what is termed the Topological Cyclotron Langmuir wave (TCLW), extensively studied in [12], and has a topologically protected edge state as we shall see below.

Particularly important for the calculation of Chern numbers are the eigenvectors associated with these eigenvalues. Assuming that $E_z = 0$ and plugging in $n^2 = R$ we get:

$$S - n^2 = S - R = -D \Rightarrow$$

$$\begin{bmatrix} -D & iD & 0 \\ -iD & -D & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \Rightarrow$$

$$E_R = (1, -i, 0)^T$$

Similarly if $n^2 = L$ we get $S - n^2 = D$ and $E_L = (1, i, 0)^T$, which represent left- and right-circularly polarized electric field as our notation suggested above. We can use the equations $k \times E = \omega B$ and $\chi E = v$ to obtain the full eigenvectors for R- and L-modes:

$$\Psi_R = \left(i\frac{\omega_p}{\Omega - \omega_R}e_-, e_-, \frac{k}{\omega_R}\hat{z} \times e_-\right)$$

$$\Psi_L = \left(-i\frac{\omega_p}{\Omega + \omega_L}e_+, e_+, \frac{k}{\omega_L}\hat{z} \times e_+\right)$$

Here we have used the circularly polarized vectors as defined in Section 1.1 $e_\pm = (1, \pm i, 0)$. To summarize we have found eigenmodes:

$$\begin{aligned} \omega_0 &= 0 \\ \Psi_0 &= (0, 0, k) \\ \omega_1 &= \omega_p \\ \Psi_1 &= (\hat{z}, i\hat{z}, 0) \\ \omega_R^3 + \Omega\omega_R^2 - (k^2 + \omega_p^2)\omega_R - k^2\Omega &= 0 \\ \Psi_R &= \left(i\frac{\omega_p}{\Omega - \omega_R}e_-, e_-, \frac{k}{\omega_R}\hat{z} \times e_-\right) \\ \omega_L^3 - \Omega\omega_L^2 - (k^2 + \omega_p^2)\omega_L + k^2\Omega &= 0 \\ \Psi_L &= \left(-i\frac{\omega_p}{\Omega + \omega_L}e_+, e_+, \frac{k}{\omega_L}\hat{z} \times e_+\right) \end{aligned}$$

As we will see later, not only does $k_\perp = 0$ provide interesting band crossings, but this case is also essential for determining topological invariants for the system.

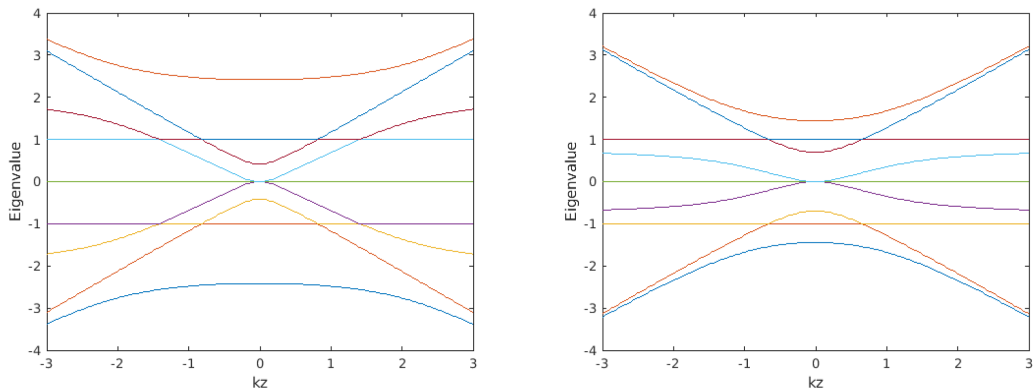

Figure 1: Eigenvalues plotted as a function of k_z for $k_\perp = 0$. All values normalized to ω_p. Left shows $\sigma = 2$ and right shows $\sigma = 0.75$. As discussed above the lower frequency L-wave starts at 0 and converges to σ as $k_z \to \infty$. The high frequency L-wave and the positive R-wave start at $\frac{1}{2}(-\sigma + \sqrt{\sigma^2 + 4})$ and $\frac{1}{2}(\sigma + \sqrt{\sigma^2 + 4})$ respectively and converge to $\omega = k_z$ asymptotically.

3.3 $k_z = 0$ Case

Like $\Omega = 0$, the $k_z = 0$ case produces coincident bands, and so is also interesting to study on its own. Calculation of eigenvectors utilizes the plasma wave equation and largely mirrors the $k_\perp = 0$ case. Full calculations are shown in Appendix B, and the eigenvalues and eigenvectors are as follows:

$$\omega_0 = 0,\ \omega_1 = \sqrt{k^2 + \omega_p^2}$$

$$\omega_2 = \sqrt{\frac{1}{2}\left(k^2 + \omega_h^2 - \sqrt{k^4 + 2(2\omega_p^2 - k^2)\Omega^2 + \Omega^4}\right)}$$

$$\omega_3 = \sqrt{\frac{1}{2}\left(k^2 + \omega_h^2 + \sqrt{k^4 + 2(2\omega_p^2 - k^2)\Omega^2 + \Omega^4}\right)}$$

$$\Psi_0^{(1)} = (0, 0, k),\ \Psi_0^{(2)} = (k \times \hat{z}, -\sigma k, i\omega_p \hat{z}),\ \Psi_0^{(3)} = (k^2 \hat{z}, 0, -ik \times \hat{z})$$

$$\Psi_1 = (-i\hat{z}, \omega_1 \hat{z}, k \times \hat{z})$$

$$\Psi_{2,3} = \left(\frac{i}{\omega_p}(I - \epsilon^{-1})k \times \hat{z},\ \epsilon^{-1}\frac{k}{\omega_{2,3}} \times \hat{z},\ -\hat{z}\right)$$

3.4 Case: $k_\perp \to \infty$

Finding eigenvectors as $k \to \infty$ will also prove to be essential for calculating Chern numbers. We will analyze this situation by breaking H into:

$$H = r \begin{bmatrix} -\frac{i\Omega}{r}\hat{z}\times & -\frac{i\omega_p}{r}I & 0 \\ \frac{i\omega_p}{r}I & 0 & -\mathbf{k}_r\times \\ 0 & \mathbf{k}_r\times & 0 \end{bmatrix} = rH_r$$

$$\mathbf{k}_r\times = \hat{k}_\perp \times + \frac{1}{r}k_z\times$$

where $\hat{k}_\perp$ is the unit vector in the $k_\perp$ direction. From this decomposition we can see that $|k_\perp| = r$ so as $r \to \infty$, $k_\perp \to \infty$. If we find the eigenvalue decomposition for H_r:

$$H_r = V_r \Lambda_r V_r^\dagger$$

then we can see that the eigenvalue decomposition for H is:

$$H = rH_r = V_r(r\Lambda_r)V_r^\dagger$$

so H and H_r share eigenvectors for all r and the eigenvalues of H are r times the eigenvalues of H_r. Since the eigenvectors are shared, we will focus on calculating the eigenvectors of H_r as $r \to \infty$.

As before, we will be concerned with the positive eigenvalues and associated eigenvectors. With this in mind, we will focus on three cases for the eigenvectors of H_r. If ω is an eigenvector of H, then as $k_\perp \to \infty$ either $\omega \to \infty$, $\omega \to \bar{\omega}$ for some $0 < \bar{\omega} < \infty$, or $\omega \to 0$. If $\omega_r = \omega/r$ is the associated eigenvalue of H_r then this corresponds to $\omega_r = c$ for some $0 < c < \infty$, $\omega_r = \bar{\omega}/r$, and $\omega_r = 0$ respectively.

First take the case that ω_r is just some positive constant. Then:

$$\begin{bmatrix} -\frac{i\Omega}{r}\hat{z}\times & -\frac{i\omega_p}{r}I & 0 \\ \frac{i\omega_p}{r}I & 0 & -\mathbf{k}_r\times \\ 0 & \mathbf{k}_r\times & 0 \end{bmatrix} \begin{bmatrix} v \\ E \\ B \end{bmatrix} = \omega_r \begin{bmatrix} v \\ E \\ B \end{bmatrix}$$

Taking $r \to \infty$ we can see from the first line that $v = 0$. The second and third lines then yield:

$$-\hat{k}_\perp \times B = \omega_r E$$

$$\hat{k}_\perp \times E = \omega_r B$$

A natural guess would be that either $E = \hat{z}$ or $B = \hat{z}$ which is perpendicular to $k_\perp$. Starting with $E = \hat{z}$ we get:

$$B = \frac{1}{\omega_r}\hat{k}_\perp \times \hat{z}$$

$$-\frac{1}{\omega_r}\hat{k}_\perp \times (\hat{k}_\perp \times \hat{z}) = \frac{1}{\omega_r}\hat{z} = \omega_r \hat{z} \Rightarrow \omega_r = 1$$

Similarly for $B = \hat{z}$:

$$-\hat{k}_\perp \times \hat{z} = \hat{z} \times \hat{k}_\perp = \omega_r E$$

$$\frac{1}{\omega_r}\hat{k}_\perp \times (\hat{z} \times \hat{k}_\perp) = \frac{\hat{z}}{\omega_r} = \omega_r \hat{z} \Rightarrow \omega_r = 1$$

Therefore $\omega_\infty = r\omega_r = k_\perp$ is an eigenvalue of H as $k_\perp \to \infty$ with eigenvectors:

$$\Psi_\infty^{(1)} = (0, \hat{z}, \hat{k}_\perp \times \hat{z})$$
$$\Psi_\infty^{(2)} = (0, \hat{k}_\perp \times \hat{z}, -\hat{z})$$

Next consider the case that $\omega_r = \bar{\omega}/r$. Plugging this in yields:

$$\begin{bmatrix} -\frac{i\Omega}{r}\hat{z}\times & -\frac{i\omega_p}{r}I & 0 \\ \frac{i\omega_p}{r}I & 0 & -\mathbf{k}_r\times \\ 0 & \mathbf{k}_r\times & 0 \end{bmatrix} \begin{bmatrix} v \\ E \\ B \end{bmatrix} = \frac{\bar{\omega}}{r} \begin{bmatrix} v \\ E \\ B \end{bmatrix}$$

Expanding the third row we get:

$$\hat{k}_\perp \times E + \frac{k_z}{r} \times E = \frac{\bar{\omega}}{r} B$$

Since the $\hat{k}_\perp \times E$ is the only term that does not go to zero we must have $E \to \hat{k}_\perp$ or $E \to 0$ as $r \to \infty$. Assuming for now that $E = \hat{k}_\perp$ from the second line we get:

$$i\frac{\omega_p}{r}v - \frac{k_z}{r} \times B - \hat{k}_\perp \times B = \frac{\bar{\omega}}{r}E$$

Again as the only term that doesn't go to zero is $-\hat{k}_\perp \times B$ we must have that $B = 0$ or $B = \hat{k}_\perp$. Now the first line give:

$$-\frac{i\Omega}{r}\hat{z} \times v - i\frac{\omega_p}{r}E = \frac{\bar{\omega}}{r}v \Rightarrow$$

$$i\Omega\hat{z} \times v - i\omega_p E = \bar{\omega} v$$

Here we will make use of the susceptibility tensor $\chi E = v$ where χ is defined with $\omega \to \bar{\omega}$. This is possible because in this form the first row is identical to the first row of (1) with $\omega \to \bar{\omega}$. Therefore:

$$v = \chi E = \frac{\omega_p}{\Omega^2 - \bar{\omega}^2} \begin{bmatrix} i\bar{\omega} & -\Omega & 0 \\ \Omega & i\bar{\omega} & 0 \\ 0 & 0 & -i(\Omega^2 - \bar{\omega}^2)/\bar{\omega} \end{bmatrix} \begin{pmatrix} \hat{k}_x \\ \hat{k}_y \\ 0 \end{pmatrix}$$

$$= \frac{\omega_p}{\Omega^2 - \bar{\omega}^2} \begin{pmatrix} i\bar{\omega}\hat{k}_x - \Omega\hat{k}_y \\ i\bar{\omega}\hat{k}_y + \Omega\hat{k}_x \\ 0 \end{pmatrix}$$

$$= \frac{\omega_p}{\Omega^2 - \bar{\omega}^2}(i\bar{\omega}\hat{k}_\perp - \Omega\hat{k}_\perp \times \hat{z})$$

Therefore we have the eigenvalue $\omega_1 = r\omega_r = \bar{\omega}$ with eigenvector:

$$\Psi_1 = \left(\frac{\omega_p}{\Omega^2 - \bar{\omega}^2} (i\bar{\omega}\hat{k}_\perp - \Omega\hat{k}_\perp \times \hat{z}), \hat{k}_\perp, 0 \right)$$

Finally we will find the zero eigenvectors. Substituting $\omega_r = 0$ and $r \to \infty$ we get from the last two lines:

$$\hat{k}_\perp \times E = 0$$

$$-\hat{k}_\perp \times B = 0$$

So again $E = 0$ or $\hat{k}_\perp$ and $B = 0$ or $\hat{k}_\perp$. From the first line we get:

$$-\Omega\hat{z} \times v = \omega_p E$$

so if $E = 0$ then $v = 0$ or $v = \hat{z}$. This recovers our universal zero eigenvector $\Psi_0 = (0, 0, \hat{k}_\perp)$. If $B = 0$ and $E = \hat{k}_\perp$ then we get:

$$-\sigma\hat{z} \times v = \hat{k}_\perp$$

To ensure a basis for the whole null space we will consider that $v = a\hat{k}_\perp \times \hat{z} \pm b\hat{z}$ so that:

$$-\sigma\hat{z} \times v = -\sigma a\hat{k}_\perp = \hat{k}_\perp \Rightarrow a = -\frac{1}{\sigma}$$

Therefore we have the zero eigenvectors:

$$\Psi_0^{(2)} = (\hat{k}_\perp \times \hat{z} + \sigma b\hat{z}, -\sigma\hat{k}_\perp, 0)$$
$$\Psi_0^{(3)} = (\hat{k}_\perp \times \hat{z} - \sigma b\hat{z}, -\sigma\hat{k}_\perp, 0)$$

Requiring that $\Psi_0^{(2)}$ and $\Psi_0^{(3)}$ are orthogonal we get:

$$\Psi_0^{*(2)}\Psi_0^{(3)} = 1 - \sigma^2 b^2 + \sigma^2 = 0 \Rightarrow$$

$$b = \sqrt{1 + \frac{1}{\sigma^2}} = \frac{\sqrt{1+\sigma^2}}{\sigma}$$

so we get three zero eigenvectors:

$$\Psi_0^{(1)} = (0, 0, \hat{k}_\perp)$$
$$\Psi_0^{(2)} = (\hat{k}_\perp \times \hat{z} + \sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{k}_\perp, 0)$$
$$\Psi_0^{(3)} = (\hat{k}_\perp \times \hat{z} - \sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{k}_\perp, 0)$$

It is also apparent now that all eigenvectors pairs are orthogonal to one another except Ψ_1 with $\Psi_0^{(2,3)}$. We also did not specify an eigenvalue for $\bar{\omega}$ however so we still have one free variable.

$$\Psi_1^*\Psi_0^{(2,3)} = \left(\frac{\omega_p}{\Omega^2 - \bar{\omega}^2} (-i\bar{\omega}\hat{k}_\perp - \Omega\hat{k}_\perp \times \hat{z}), \hat{k}_\perp, 0 \right) \cdot (\hat{k}_\perp \times \hat{z} \pm \sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{k}_\perp, 0)^T$$

$$= -\frac{\Omega\omega_p}{\Omega^2 - \bar{\omega}^2} - \sigma = 0 \Rightarrow$$
$$\bar{\omega}^2 = \omega_p^2 + \Omega^2 = \omega_{uh}^2$$

Plugging this result in and summarizing, the eigenvectors and eigenvalues of H as $k_\perp \to \infty$ are:

$$\omega_0 = 0$$
$$\Psi_0^{(1)} = (0, 0, \hat{k}_\perp) \;\; \Psi_0^{(2)} = (\hat{k}_\perp \times \hat{z} + \sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{k}_\perp, 0) \;\; \Psi_0^{(3)} = (\hat{k}_\perp \times \hat{z} - \sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{k}_\perp, 0)$$
$$\omega_1 = \omega_{uh}$$
$$\Psi_1 = \left(\sigma\hat{k}_\perp \times \hat{z} - i\frac{\omega_{uh}}{\omega_p}\hat{k}_\perp, \hat{k}_\perp, 0\right)$$
$$\omega_2 = k_\perp (\to \infty)$$
$$\Psi_2^{(1)} = (0, \hat{z}, \hat{k}_\perp \times \hat{z}) \;\; \Psi_2^{(2)} = (0, \hat{k}_\perp \times \hat{z}, -\hat{z})$$

4 Chern Numbers

The Chern number of a particular level of the system is defined as the integral over the parameter space (in our case $\mathbf{k}_\perp$) of the Berry Curvature:

$$C_n = \frac{1}{2\pi}\int F_n(\mathbf{k}_\perp) \cdot d\mathbf{k}_\perp \tag{13}$$

The Berry curvature in a 2-d parameter space is defined as the curl of the Berry connection $A(\mathbf{k}_\perp)$:

$$A_n(\mathbf{k}_\perp) = i\langle\Psi_n|\nabla_{k_\perp}\Psi_n\rangle$$
$$F_n(\mathbf{k}_\perp) = \nabla_{\mathbf{k}_\perp} \times A_n(\mathbf{k}_\perp) \tag{14}$$

Plugging this definition in results in the more compact form for the Berry curvature:

$$F_n(\mathbf{k}_\perp) = -2\ \mathrm{Im}\langle\partial_{k_x}\Psi_n|\partial_{k_y}\Psi_n\rangle \tag{15}$$

We can also use Stokes theorem to derive the following definition, which is especially useful in the continuum case [14]:

$$\int_S \nabla_{\mathbf{k}_\perp} \times A_n(\mathbf{k}_\perp) \cdot d\mathbf{k}_\perp = \oint_{\partial S} A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} \Rightarrow$$
$$C_n = \frac{1}{2\pi}\left[\oint_{k\to\infty} A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} - \oint_{k\to 0} A_n(\mathbf{k}_\perp) \cdot \mathbf{dl}\right] \tag{16}$$

Finally, assuming that our eigenvectors form an orthonormal basis we have another form of the Berry Curvature which avoids taking derivatives of the eigenvectors themselves:

$$F_n(\mathbf{k}) = -2\mathrm{Im}\left[\sum_{m\neq n} \frac{\langle\Psi_n|\partial_{k_x}H|\Psi_m\rangle\langle\Psi_m|\partial_{k_y}H|\Psi_n\rangle}{(\omega_n - \omega_m)^2}\right] \tag{17}$$

For full derivation of the above quantities and deeper analysis and discussion see [9]. When the dependence of the eigenvectors on $\mathbf{k}_\perp$ is straightforward ($\Omega = 0$) using (13) and (15) is is sufficient. For more complex dependence of Ψ_n on $\mathbf{k}_\perp$ (all other cases) (16) will be more useful, and for numerical calculations (17) is the most useful (having transferred the derivatives to H allows numerical calculation of Ψ_n's).

Next, by leveraging the k_x/k_y plane symmetry, we can find a more useful form of (16). From Section 3.4 and (17) we can clearly see (since $R^T R = I$) that the Berry Curvature in our case is rotationally symmetric in the x/y plane. We can also use the rotational symmetry of eigenvectors in the k_x/k_y plane to derive a more direct equation for C_n using (16). First, parametrize the $\mathbf{k}_\perp$ plane in terms of (k, θ), radial and polar coordinates in the $\mathbf{k}_\perp$ plane:

$$k_x = k\cos\theta$$

$$k_y = k\sin\theta$$

Then we can write the contour integral in (16) as:

$$\oint A_n(\mathbf{k}_\perp)\cdot\mathbf{dl} = i\oint \Psi_n^* D_{\mathbf{k}_\perp}\Psi_n\cdot\mathbf{dl} = ik\int_0^{2\pi}(\Psi_n^* D_{\mathbf{k}_\perp}\Psi_n)\cdot\hat{\theta}\,d\theta$$

$$= ik\int_0^{2\pi}\begin{pmatrix}\Psi_n^*\partial_{k_x}\Psi_n, & \Psi_n^*\partial_{k_y}\Psi_n\end{pmatrix}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}d\theta$$

$$= ik\int_0^{2\pi}\cos\theta(\Psi_n^*\partial_{k_y}\Psi_n) - \sin\theta(\Psi_n^*\partial_{k_x}\Psi_n)d\theta$$

We can calculate:

$$\partial_{k_x}\Psi_n = \frac{\partial k}{\partial k_x}\partial_k\Psi_n + \frac{\partial\theta}{\partial k_x}\partial_\theta\Psi_n = \frac{k\cos\theta\partial_k\Psi_n - \sin\theta\partial_\theta\Psi_n}{k}$$

$$\partial_{k_y}\Psi_n = \frac{\partial k}{\partial k_y}\partial_k\Psi_n + \frac{\partial\theta}{\partial k_y}\partial_\theta\Psi_n = \frac{k\sin\theta\partial_k\Psi_n + \cos\theta\partial_\theta\Psi_n}{k}$$

$$\Rightarrow k\left(\cos\theta(\Psi_n^*\partial_{k_y}\Psi_n) - \sin\theta(\Psi_n^*\partial_{k_x}\Psi_n)\right)$$

$$= -k\cos\theta\sin\theta\Psi_n^*\partial_k\Psi_n + \sin^2\theta\Psi_n^*\partial_\theta\Psi_n + k\cos\theta\sin\theta\Psi_n^*\partial_k\Psi_n + \cos^2\theta\Psi_n^*\partial_\theta\Psi_n$$

$$= \Psi_n^*\partial_\theta\Psi_n$$

Therefore we get:

$$\oint A_n(\mathbf{k}_\perp)\cdot\mathbf{dl} = ik\int_0^{2\pi}\cos\theta(\Psi_n^*\partial_{k_y}\Psi_n) - \sin\theta(\Psi_n^*\partial_{k_x}\Psi_n)d\theta = i\int_0^{2\pi}\Psi_n^*\partial_\theta\Psi_n d\theta$$

Now due to the rotational symmetry of Ψ_n we can parametrize Ψ_n as $\Psi_n = \mathbf{R}\Psi_{n0}$ where $\Psi_{n0} = \Psi_n(k, \theta = 0)$ and $\mathbf{R}$ defined as in Section 3.4. Then since Ψ_{n0} is fixed we get:

$$\Psi_n^*\partial_\theta\Psi_n = \Psi_{n0}^*\mathbf{R}^T\partial_\theta(\mathbf{R}\Psi_{n0}) = \Psi_{n0}^*\mathbf{R}^T\partial_\theta(\mathbf{R})\Psi_{n0}$$

Remembering from Section 3.4 that:

$$\mathbf{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we get that

$$\partial_\theta \mathbf{R} = \begin{bmatrix} \partial_\theta R & 0 & 0 \\ 0 & \partial_\theta R & 0 \\ 0 & 0 & \partial_\theta R \end{bmatrix}$$

with

$$\partial_\theta R = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R = [\hat{z}\times]R = R\hat{z}\times$$

where in the last equality we used the fact from 3.4 that R and $\hat{z}\times$ commute. Denoting the matrix:

$$\mathbf{Z}\times = \begin{bmatrix} \hat{z}\times & 0 & 0 \\ 0 & \hat{z}\times & 0 \\ 0 & 0 & \hat{z}\times \end{bmatrix}$$

we get:

$$\Psi_n^* \partial_\theta \Psi_n = \Psi_{n0}^* \mathbf{R}^T \mathbf{R} \mathbf{Z} \times \Psi_{n0} = \Psi_{n0}^* \mathbf{Z} \times \Psi_{n0}$$

Therefore we get:

$$\oint A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} = i \int_0^{2\pi} \Psi_n^* \partial_\theta \Psi_n d\theta = i \int_0^{2\pi} \Psi_{n0}^* \mathbf{Z} \times \Psi_{n0} d\theta$$

Since Ψ_{n0} is fixed for a given k we can plug this into (16) to get:

$$C_n = \lim_{k\to\infty} i\Psi_{n0}^* (\mathbf{Z} \times \Psi_{n0}) - \lim_{k\to 0} i\Psi_{n0}^* (\mathbf{Z} \times \Psi_{n0}) \tag{18}$$

Note that this result holds for any 2-dimensional parameter space with rotational symmetry (e.g. any system where the final result of 3.4 holds).

4.1 Chern Number Symmetry

Suppose that we have calculated a Chern number C_n for a positive band with eigenvalue ω_n with positive Ω. Supposing that the eigenvector for this band is simply $(v, E, B)^T$, we know that the Chern number is given by (16) with:

$$\Psi_{n0}^* \mathbf{Z} \times \Psi_{n0} = v^\dagger(\hat{z} \times v) + E^\dagger(\hat{z} \times E) + B^\dagger(\hat{z} \times B)$$

Now suppose that we want to find the Chern number C_{-n} for the associated negative band $-\omega_n$. From Section 4.1 we know the associated eigenvector is $(v^*, E^*, -B^*)$ so we get:

$$\Psi^*_{-n0}\mathbf{Z}\times\Psi_{-n0} = v^T(\hat{z}\times v^*)+E^T(\hat{z}\times E^*)+B^T(\hat{z}\times B^*) = (v^\dagger(\hat{z}\times v))^*+(E^\dagger(\hat{z}\times E))^*+(B^\dagger(\hat{z}\times B))^*$$

$$= (\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0})^*$$

Assuming that C_n is a real number, we know that $\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0}$ must be purely imaginary, so we get:

$$\Psi^*_{-n0}\mathbf{Z}\times\Psi_{-n0} = (\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0})^* = -\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0}$$

$$\Rightarrow C_{-n} = -C_n$$

from (16). If we repeat this analysis instead for $C_{-\Omega n}$, the Chern number the ω_n band reflected across $\Omega = 0$ we get an identical result by using the results of Section 4.2, so $C_{-\Omega n} = -C_n$. Applying the results of Section 4.3 and 4.4 it's clear that reflecting $k_z \to -k_z$ gives an identical Chern number, $C_{-k_z n} = C_n$. Therefore, it's sufficient to calculate Chern numbers for bands that have $\omega, \Omega, k_z > 0$.

4.2 Chern Number Calculations

We now have all the tools necessary to calculate Chern numbers for all bands. First the general case is presented, but the $B = 0$ and $k_z = 0$ are also interesting to consider by themselves and are shown as well.

4.2.1 General Chern Numbers

Here we calculate Chern numbers for non-negative bands assuming that $\Omega, k_z > 0$. The symmetry derived in Section 6.1 generalizes these Chern numbers to any $\Omega, k_z \neq 0$. To use (18) we need only calculate 8 values, two for each positive eigenvalue ω_n, to calculate C_n:

$$\lim_{k_\perp\to 0}\frac{i\Psi^*_n(\mathbf{Z}\times\Psi_n)}{|\Psi_n|^2}$$

and

$$\lim_{k_\perp\to \infty}\frac{i\Psi^*_n(\mathbf{Z}\times\Psi_n)}{|\Psi_n|^2}$$

for $n \in \{1, 2, 3, 4\}$. It's easy to see that:

$$i\Psi^*_0(\mathbf{Z}\times\Psi_0) = i(0,0,k)\mathbf{Z}\times\begin{pmatrix}0\\0\\k\end{pmatrix} = ik\cdot\hat{z}\times k = 0$$

both as $k_\perp \to 0$ and $k_\perp \to \infty$ so $C_0 = 0$. We have already computed the eigenvectors for $k_\perp = 0$ and as $k_\perp \to \infty$ so for the positive bands we can simply substitute our results from Sections 5.2 and 5.4 respectively. First for $k_\perp = 0$ we get:

$$i\Psi_p^*(\mathbf{Z}\times\Psi_p) = i(\hat{z}, -i\hat{z}, 0)\begin{pmatrix}\hat{z}\times\hat{z}\\ i\hat{z}\times\hat{z}\\ 0\end{pmatrix} = 0$$

$$i\Psi_R^*(\mathbf{Z}\times\Psi_R) = i\left(-i\frac{\omega_p}{\Omega-\omega_R}e_-^\dagger, e_-^\dagger, \frac{k}{\omega_R}(\hat{z}\times e_-)^\dagger\right)\begin{pmatrix} i\frac{\omega_p}{\Omega-\omega_R}\hat{z}\times e_-\\ \hat{z}\times e_-\\ -\frac{k}{\omega_R}e_-\end{pmatrix}$$

$$= i\left(\left(1+\frac{\omega_p^2}{(\Omega-\omega_R)^2}\right)e_-^\dagger(\hat{z}\times e_-) - \frac{k^2}{\omega_R^2}(\hat{z}\times e_-)^\dagger e_-\right)$$

We can also evaluate:

$$e_-^\dagger(\hat{z}\times e_-) = (1, i, 0)\begin{pmatrix} i\\ 1\\ 0\end{pmatrix} = 2i$$

$$(\hat{z}\times e_+)^\dagger e_+ = (e_+^\dagger(\hat{z}\times e_+))^* = -2i$$

Therefore:

$$i\Psi_R^*(\mathbf{Z}\times\Psi_R) = 2\left(\left(1+\frac{\omega_p^2}{(\Omega-\omega_R)^2}\right) + \frac{k^2}{\omega_R^2}\right)$$

Also notice that

$$e_-^\dagger e_- = 2$$

so we have:

$$|\Psi_R|^2 = 2\left(\left(1+\frac{\omega_p^2}{(\Omega-\omega_R)^2}\right) + \frac{k^2}{\omega_R^2}\right)$$

Therefore:

$$\frac{i\Psi_R^*(\mathbf{Z}\times\Psi_R)}{|\Psi_R|^2} = 1$$

Finally for ω_L we get:

$$i\Psi_L^*\mathbf{Z}\times\Psi_L = i\left(i\frac{\omega_p}{\Omega+\omega_L}e_+^\dagger, e_+^\dagger, \frac{k}{\omega_L}(\hat{z}\times e_+)^\dagger\right)\begin{pmatrix} -i\frac{\omega_p}{\Omega+\omega_L}\hat{z}\times e_+\\ \hat{z}\times e_+\\ -\frac{k}{\omega_L}e_+\end{pmatrix}$$

$$= i\left(\left(1+\frac{\omega_p^2}{(\Omega+\omega_L)^2}\right)e_+^\dagger(\hat{z}\times e_+) - \frac{k^2}{\omega_L^2}(\hat{z}\times e_+)^\dagger e_+\right)$$

Once again calculate:

$$e_+^\dagger(\hat{z}\times e_+) = (1, -i, 0)\begin{pmatrix} -i\\ 1\\ 0\end{pmatrix} = -2i$$

$$(\hat{z} \times e_+)^\dagger e_- = (e_+^\dagger(\hat{z} \times e_-))^* = +2i$$

Therefore:

$$i\Psi_L^* \mathbf{Z} \times \Psi_L = 2\left(\left(1 + \frac{\omega_p^2}{(\Omega + \omega_L)^2}\right) + \frac{k^2}{\omega_L^2}\right)$$

We can also see that $e_-^\dagger e_- = 2$ so that:

$$\frac{i\Psi_L^* \mathbf{Z} \times \Psi_L}{|\Psi_L|^2} = +1$$

Now for the $k_\perp \to \infty$ case. We can see that all the eigenvectors except Ψ_1 are real-valued. It's straightforward to verify that if $\Psi \in \mathbb{R}^9$:

$$\Psi^* \mathbf{Z} \times \Psi = \Psi^T \mathbf{Z} \times \Psi = 0$$

Now consider Ψ_1:

$$\Psi_1^* \mathbf{Z} \times \Psi_1 = i\left(i\frac{\omega_{uh}}{\omega_p}\hat{k}_\perp + \sigma \hat{k}_\perp \times \hat{z}, \hat{k}_\perp, 0\right)\begin{pmatrix} i\frac{\omega_{uh}}{\omega_p}\hat{k}_\perp \times \hat{z} + \sigma \hat{k}_\perp \\ \hat{k}_\perp \times \hat{z} \\ 0 \end{pmatrix}$$

$$= 2\sigma\frac{\omega_{uh}}{\omega_p} = 2\sigma\sqrt{1+\sigma^2}$$

We can also calculate:

$$|\Psi_1|^2 = \frac{\omega_{uh}^2}{\omega_p^2} + \sigma^2 + 1 = \frac{\Omega^2 + \omega_p^2}{\omega_p^2} + \sigma^2 + 1 = 2(\sigma^2 + 1)$$

Therefore:

$$\frac{\Psi_1^* \mathbf{Z} \times \Psi_1}{|\Psi_1|^2} = \frac{2\sigma\sqrt{1+\sigma^2}}{2(1+\sigma^2)} = \frac{\sigma}{\sqrt{1+\sigma^2}}$$

Now we have all 8 quantities needed to calculate all Chern numbers, but it is unclear which $k_\perp = 0$ eigenvectors match with which $k_\perp \to \infty$ eigenvectors. As $k_\perp \to \infty$ it's clear that $0 = \omega_0 < \omega_1 = \omega_{uh} < \omega_2 = k_\perp$. We will assume for now and verify later that the bands only cross when $k_\perp = 0$, $\Omega = 0$, or $k_z = 0$ and in addition that when $k_\perp = 0$ the R and L bands never cross one another. Since we have assumed in this section that $\Omega, k_z > 0$ this means that the order of eigenvalues at $k_\perp = 0$ determines the order at all intermediate values of $k_\perp$. Therefore, determining the order of eigenvalues at $k_\perp = 0$ allows us to pair eigenvectors at $k_\perp = 0$ with eigenvectors at $k_\perp \to \infty$.

For the purposes of this section we will call the lower (Cyclotron) L-mode ω_{L^-} and the upper L-mode ω_{L^+}. Assuming that the L and R-bands never cross we know from our analysis at $k_z = 0$ that $\omega_L^- < \omega_R < \omega_{L^+}$ From our analysis in Section 5.2 we found two critical values of ω_p:

$$\omega_\pm = \frac{\sqrt{k_z^4 + 4\Omega^2 k_z^2} \pm k_z^2}{2\Omega}$$

where ω_- only exists if $\Omega > \omega_p$. Clearly $\omega_- < \omega_+$. These boundaries allow us to form three topologically distinct regions in parameter space. If $\omega_- < \omega_+ < \omega_p$ we know that $\omega_{L^-} < \omega_R < \omega_p$. Following the convention in [2] we will name this region III. If instead $\omega_- < \omega_p < \omega_+$ we have $\omega_{L^-} < \omega_p < \omega_R$, which is termed region II. Finally if $\omega_p < \omega_- < \omega_+$ we have $\omega_p < \omega_{L_-} < \omega_R$, which is termed region I. This allows us to pair $k_\perp = 0$ eigenvectors with $k_\perp \to \infty$ eigenvectors and calculate Chern numbers as follows:

Region	Eigenvector Pairing	Chern Numbers (C_1, C_2, C_3, C_4)
I	$\Psi_p \to \Psi_0^{(2)}$, $\Psi_{L^-} \to \Psi_1$, $\Psi_R \to \Psi_2^{(1)}$, $\Psi_{L^+} \to \Psi_2^{(2)}$	$\left(0, \frac{\sigma}{\sqrt{\sigma^2+1}} - 1, 1, -1\right)$
II	$\Psi_{L^-} \to \Psi_0^{(2)}$, $\Psi_p \to \Psi_1$, $\Psi_R \to \Psi_2^{(1)}$, $\Psi_{L^+} \to \Psi_2^{(2)}$	$\left(-1, \frac{\sigma}{\sqrt{\sigma^2+1}}, 1, -1\right)$
III	$\Psi_{L^-} \to \Psi_0^{(2)}$, $\Psi_R \to \Psi_1$, $\Psi_p \to \Psi_2^{(1)}$, $\Psi_{L^+} \to \Psi_2^{(2)}$	$\left(-1, 1 + \frac{\sigma}{\sqrt{\sigma^2+1}}, 0, -1\right)$

4.2.2 $B_0 = 0$

Due to the straightforward expressions for eigenvectors in the $\Omega = 0$ case we can apply (14) directly to calculate Chern numbers in this case. Detailed calculations are shown in Appendix A and result in all trivial Chern numbers in this case $C_n = 0$.

4.2.3 $k_z = 0$

We shall see below that $k_z = 0$, although nominally part of region III calculated above, is topologically distinct from the general case. For an alternate approach of $k_z = 0$ Chern number calculations see [5], where these quantities were first calculated. To summarize the results of Section 4.3 we have the positive eigenvalues and associated eigenvectors of the $k_z = 0$ case below:

$$\omega_0 = 0,\ \omega_1 = \sqrt{k^2 + \omega_p^2}$$

$$\omega_2 = \sqrt{\frac{1}{2}\left(k^2 + \omega_h^2 - \sqrt{k^4 + 2(2\omega_p^2 - k^2)\Omega^2 + \Omega^4}\right)}$$

$$\omega_3 = \sqrt{\frac{1}{2}\left(k^2 + \omega_h^2 + \sqrt{k^4 + 2(2\omega_p^2 - k^2)\Omega^2 + \Omega^4}\right)}$$

$$\Psi_0^{(1)} = (0, 0, k),\ \Psi_0^{(2)} = (k \times \hat{z}, -\sigma k, i\omega_p \hat{z}),\ \Psi_0^{(3)} = (k^2 \hat{z}, 0, -ik \times \hat{z})$$

$$\Psi_1 = (-i\hat{z}, \omega_1 \hat{z}, k \times \hat{z})$$

$$\Psi_{2,3} = \left(\frac{i}{\omega_p}(I - \epsilon^{-1})k \times \hat{z},\ \epsilon^{-1}\frac{k}{\omega_{2,3}} \times \hat{z},\ -\hat{z}\right)$$

Using (15) we can calculate as in the last section:

$$\partial_{k_x}\Psi_0^{(1)} = (0, 0, \hat{x}) \quad \partial_{k_x}\Psi_0^{(2)} = (-\hat{y}, \sigma\hat{x}, 0) \quad \partial_{k_x}\Psi_0^{(3)} = (2k_x\hat{z}, 0, -i\hat{y})$$
$$\partial_{k_y}\Psi_0^{(1)} = (0, 0, \hat{y}) \quad \partial_{k_y}\Psi_0^{(2)} = (\hat{x}, \sigma\hat{y}, 0) \quad \partial_{k_y}\Psi_0^{(3)} = (2k_y\hat{z}, 0, i\hat{x})$$

$$\Rightarrow (\partial_{k_x}\Psi_0^*)\partial_{k_y}\Psi_0 = 0$$

for all values of k, Therefore $C_0 = 0$. For Ψ_1 we have:

$$\partial_{k_x}\Psi_1 = (0, \frac{2|k|k_x}{\omega_1}\hat{z}, -\hat{y})$$

$$\partial_{k_y}\Psi_1 = (0, \frac{2|k|k_y}{\omega_1}\hat{z}, \hat{x})$$

$$F_1(k) = -2\mathrm{Im}\left(\frac{4k^2 k_x k_y}{\omega_1^2}\right) = 0$$

$$\Rightarrow C_1 = 0$$

Finally, for the transverse magnetic modes $\Psi_2,\ \Psi_3$ the dependence on k is rather complex- Ψ_2 and Ψ_3 depend on $\omega_{2/3}$ and ϵ^{-1}, which itself depends on k and ω. Therefore (15) is not tractable and using (18) is more appropriate. First, for the transverse magnetic modes it is necessary to calculate ϵ^{-1}. Remembering that:

$$\epsilon = \begin{bmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

with S, D, and P as defined in Section 4.2 we get:

$$\epsilon^{-1} = \frac{1}{S^2 - D^2}\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & \frac{S^2-D^2}{P} \end{bmatrix}$$

Comparing with the calculations from Section 4.3 we can see that

$$\frac{S}{S^2 - D^2} = \frac{S}{RL} = \frac{\omega^2}{k^2} = \frac{\omega^2(\omega^2 - \omega_{uh}^2)}{\omega^4 - 2\omega_h^2\omega^2 + \omega_p^2}$$

Similarly:

$$\frac{D}{RL} = \frac{\omega_p^2(\omega + \Omega) - \omega_p^2(\omega - \Omega)}{\omega(\omega^2 - \Omega^2)}\frac{\omega^2(\omega^2 - \Omega^2)}{\omega^4 - 2\omega_h^2\omega^2 + \omega_p^2} = \frac{\omega_p^2\Omega\omega}{\omega^4 - 2\omega_h^2\omega^2 + \omega_p^2} = \frac{\omega^2}{k^2}\frac{\omega_p^2\Omega}{\omega(\omega^2 - \omega_{uh}^2)}$$

Plugging these results in we get:

$$\epsilon^{-1} = \frac{\omega^2}{k^2}\begin{bmatrix} 1 & -i\frac{\Omega\omega_p^2}{\omega(\omega^2-\omega_{uh}^2)} & 0 \\ i\frac{\Omega\omega_p^2}{\omega(\omega^2-\omega_{uh}^2)} & 1 & 0 \\ 0 & 0 & \frac{k^2}{\omega^2-\omega_p^2} \end{bmatrix}$$

Define more compact variables α and β for the following calculations we get:

$$\epsilon^{-1} = \begin{bmatrix} \alpha & -i\beta & 0 \\ i\beta & \alpha & 0 \\ 0 & 0 & 1/P \end{bmatrix}$$

$$\alpha = \frac{\omega^2}{k^2} = \frac{(\omega^2 - \omega_{uh}^2)\omega^2}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4}$$

$$\beta = \frac{\omega_p^2 \Omega \omega}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4} = \frac{\alpha\omega_p^2\Omega}{\omega(\omega^2 - \omega_{uh}^2)}$$

This allows us to explicitly calculate E and v for the transverse magnetic waves:

$$E = \epsilon^{-1}\frac{k}{\omega}\times\hat{z} = \frac{1}{\omega}\begin{bmatrix} \alpha & -i\beta & 0 \\ i\beta & \alpha & 0 \\ 0 & 0 & 1/P \end{bmatrix}\begin{bmatrix} k_y \\ -k_x \\ 0 \end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} i\beta k_x - \alpha k_y \\ i\beta k_y - \alpha k_x \\ 0 \end{bmatrix} = \frac{1}{\omega}(i\beta k\alpha - k\times\hat{z})$$

$$E^* = -\frac{1}{\omega}(\alpha k \times \hat{z} i\beta k)$$

$$\hat{z} \times E = \frac{1}{\omega}(\alpha\hat{z} \times (k \times \hat{z}) - i\beta\hat{z} \times k) = \frac{1}{\omega}(\alpha k - i\beta\hat{z} \times k)$$

$$v = \frac{i}{\omega_p}(I - \epsilon^{-1})k\times\hat{z} = \frac{i}{\omega_p}(k\times\hat{z} - \epsilon^{-1}k\times\hat{z}) = \frac{i}{\omega_p}(k\times\hat{z} - \omega E) = \frac{i}{\omega_p}(k\times\hat{z} - (\alpha k\times\hat{z} - i\beta k)) = i\frac{(1-\alpha)}{\omega_p}k\times\hat{z} - \frac{\beta}{\omega_p}k$$

$$v^* = -i\frac{(1-\alpha)}{\omega_p}k \times \hat{z} - \frac{\beta}{\omega_p}k$$

$$\hat{z} \times v = i\frac{(1-\alpha)}{\omega_p}k - \frac{\beta}{\omega_p}\hat{z} \times k$$

Calculating the Chern number from the Berry Connection as in (18) assumes normalized eigenvectors. We have not normalized our eigenvectors in this case and so will need to do so. Therefore our quantity of interest is:

$$\frac{i\Psi^*(\mathbf{Z} \times \Psi)}{|\Psi|^2} = i\frac{v^*(\hat{z} \times v) + E^*(\hat{z} \times E) + B^*(\hat{z} \times B)}{|v|^2 + |E|^2 + |B|^2} = i\frac{-i\frac{2\beta(1-\alpha)k^2}{\omega_p^2} + i\frac{2\alpha\beta k^2}{\omega^2}}{\frac{k^2}{\omega_p^2}((1-\alpha)^2 + \beta^2) + \frac{1}{\alpha}(\alpha^2 + \beta^2) + 1} =$$

$$\frac{-2\beta(\omega_p^2 + (\alpha - 1)k^2)}{\omega_p^2 + k^2((1-\alpha)^2 + \beta^2) + \frac{\omega_p^2}{\alpha}(\beta^2 + \alpha^2)}$$

First consider $k \to \infty$. For the upper transverse magnetic mode (ω_3) we saw in Section 4.3 that as $k \to \infty$, $\omega_3 \to \infty$, and more precisely $\omega_3 \to k$. Plugging this result into the definition of α and β we get that $\lim_{k\to\infty}\alpha = 1$ and $\lim_{k\to\infty}\beta = 0$. Therefore for the upper TM mode:

$$\lim_{k\to\infty}\frac{-2\beta(\omega_p^2 + (\alpha - 1)k^2)}{\omega_p^2 + k^2((1-\alpha)^2 + \beta^2) + \frac{\omega_p^2}{\alpha}(\beta^2 + \alpha^2)} = \lim_{k\to\infty}\frac{-2\beta(\frac{\omega_p^2}{k^2} + (\alpha - 1))}{\frac{\omega_p^2}{k^2} + ((1-\alpha)^2 + \beta^2) + \frac{\omega_p^2}{k^2\alpha}(\beta^2 + \alpha^2)}$$

$$\lim_{k\to\infty}\frac{-2\omega^2\beta(\frac{\omega_p^2}{k^2} + (\alpha - 1))}{\alpha\omega_p^2 + \omega^2((1-\alpha^2) + \beta^2) + \omega_p^2(\beta^2 + \alpha^2)} = 0$$

It is not immediately apparent that $\omega^2\beta \not\to \infty$ but we can calculate:

$$\omega^2\beta = \alpha\beta k^2 = \alpha^2 \frac{k^2\omega_p^2\Omega}{\omega(\omega^2 - \omega_{uh}^2)} \to 0$$

since $\omega^2 \to k^2$.

For the lower TM mode (ω_2) we calculated in Section 4.3 that as $k \to \infty$ $\omega_2 \to \omega_{uh}$. It's clear from the definition of α that in this case $\alpha \to 0$. For β we get:

$$\lim_{k\to\infty} \beta = \frac{\omega_p^2\Omega\omega_{uh}}{\omega_{uh}^4 - \omega_{uh}^2\omega_h^2 + \omega_p^4} = \frac{\omega_p^2\Omega\omega_{uh}}{\omega_p^4 + \Omega^4 + 2\omega_p^2\Omega^2 - (\omega_p^2 + \Omega^2)(2\omega_p^2 + \Omega^2) + \omega_p^4} = -\frac{\omega_{uh}}{\Omega}$$

Plugging this in we get:

$$\lim_{k\to\infty} \frac{i\Psi_2^*(\mathbf{Z} \times \Psi_2)}{|\Psi_2|^2} = \lim_{k\to\infty} \frac{-2\beta(\frac{\omega_p^2}{k^2} + (\alpha - 1))}{\frac{\omega_p^2}{k^2} + ((1-\alpha)^2 + \beta^2) + \frac{\omega_p^2}{k^2\alpha}(\beta^2 + \alpha^2)} = \frac{-2\omega_{uh}}{\Omega + \frac{\omega_{uh}^2}{\Omega} + \frac{\omega_p^2\omega_{uh}^2}{\Omega\omega_{uh}^2}}$$

$$= -\frac{2\omega_{uh}}{2(\Omega + \frac{\omega_p^2}{\Omega})} = -\frac{\Omega\omega_{uh}}{\omega_{uh}^2} = -\frac{\Omega}{\omega_p\sqrt{1 + \Omega^2/\omega_p^2}} = -\frac{\sigma}{\sqrt{1+\sigma^2}}$$

Now as $k \to 0$ we get that:

$$\omega_2^2 \to \frac{\omega_h^2}{2} - \frac{1}{2}\sqrt{4\omega_p^2\Omega^2 + \Omega^4} = \omega_{0^-}^2$$

and:

$$\omega_3 \to \frac{\omega_h^2}{2} + \frac{1}{2}\sqrt{4\omega_p^2\Omega^2 + \Omega^4} = \omega_{0^+}^2$$

It will also be useful to define $y = \frac{1}{2}\sqrt{4\omega_p^2\Omega^2 + \Omega^4}$. Looking closely we can see that these are actually the zeros of $\omega^4 - \omega^2\omega_h^2 + \omega_p^4$ so $\alpha, \beta \to \infty$ as $k \to 0$. Therefore we can calculate:

$$\lim_{k\to 0} \frac{i\Psi_2^*(\mathbf{Z} \times \Psi_2)}{|\Psi_2|^2} = \frac{-2(\omega_p^2 + \alpha k^2 - k^2)}{\frac{\omega_p^2}{\beta} + \frac{k^2}{\beta}((1-\alpha)^2 + \beta^2) + \frac{\omega_p^2}{\alpha\beta}(\alpha^2 + \beta^2)}$$

$$= \frac{-2(\omega_p^2 + \omega_{0^-}^2)}{\frac{\omega_p^2}{\beta} + \frac{\omega_{0^-}^2\alpha}{\beta} - 2\frac{\omega_{0^-}^2}{\beta} + \frac{k^2}{\beta} + k^2\beta + \omega_p^2(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} = \frac{-2(\omega_p^2 + \omega_{0^-}^2)}{(\omega_p^2 + \omega_{0^-}^2)(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})} = \frac{-2}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}}$$

Here we have made use of the fact that $\alpha k^2 = \omega^2$ and $\beta k^2 = \beta\omega^2/\alpha$. Now we must find $\lim_{k\to 0} \alpha/\beta$:

$$\lim_{k\to 0} \frac{\alpha}{\beta} = \frac{\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)}{\omega_p^2\Omega}$$

We will first show that $\omega_{0^-}^2(\omega_{0^-}^2 - \omega_{uh}^2)^2 = \omega_p^4\Omega^2$. Plugging in definitions we get:

$$\omega_{0^-}^2 - \omega_{uh}^2 = \omega_p^2 + \frac{\Omega^2}{2} - y - \omega_p^2 - \Omega^2 = -\frac{\Omega^2}{2} - y$$

$$\Rightarrow (\omega_{0^-}^2 - \omega_{uh}^2)^2 = \frac{\Omega^4}{4} + y^2 + \Omega^2 y = \frac{\Omega^4}{4} + \omega_p^2\Omega^2 + \frac{\Omega^4}{4} + \Omega^2 y = \Omega^2(\frac{\omega_h^2}{2} + y)$$

Therefore we get:

$$\omega_{0^-}^2(\omega_{0^-}^2 - \omega_{uh}^2)^2 = \Omega^2(\frac{\omega_h^2}{2}+y)(\frac{\omega_h^2}{2}-y) = \Omega^2(\frac{\omega_h^4}{4}-y^2) = \Omega^2(\frac{4\omega_p^4 + 4\Omega^2\omega_p^2 + \Omega^4}{4} - (\omega_p^2\Omega^2 + \frac{\Omega^4}{4})) = \omega_p^4\Omega^2$$

$$\Rightarrow \omega_p^2\Omega = \pm\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)$$

From Section 4.3 we know that $\omega_{0^-} \leq \omega^2 \leq \omega_{uh}$. Therefore, assuming for now that $\Omega > 0$ we have that:

$$\omega_p^2\Omega = -\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)$$

since $\omega_p^2\Omega > 0$ and $\omega_{0^-}^2 - \omega_{uh}^2 < 0$. Therefore for the lower TM mode we get:

$$\lim_{k\to 0}\frac{\alpha}{\beta} = \lim_{k\to 0}\frac{\beta}{\alpha} = \frac{\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)}{\omega_p^2\Omega} = -1$$

This gives:

$$\lim_{k\to 0}\frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} = \frac{-2}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}} = -1$$

Repeating the above analysis with $\omega \to \omega_{0^+}$ gives identical results except that $\omega_{0^+} > \omega_{uh}$ so:

$$\omega_p^2\Omega = \omega_{0^+}(\omega_{0^+}^2 - \omega_{uh}^2)$$

Therefore:

$$\lim_{k\to 0}\frac{\alpha}{\beta} = \lim_{k\to 0}\frac{\beta}{\alpha} = \frac{\omega_{0^+}(\omega_{0^+}^2 - \omega_{uh}^2)}{\omega_p^2\Omega} = 1$$

and we get:

$$\lim_{k\to 0}\frac{i\Psi_3^*(\mathbf{Z}\times\Psi_3)}{|\Psi_3|^2} = \frac{-2}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}} = 1$$

Finally, plugging these results into (18) gives:

$$C_3 = \lim_{k\to\infty}\frac{i\Psi_3^*(\mathbf{Z}\times\Psi_3)}{|\Psi_3|^2} - \lim_{k\to 0}\frac{i\Psi_3^*(\mathbf{Z}\times\Psi_3)}{|\Psi_3|^2} = -1$$

and for the lower harmonic:

$$C_2 = \lim_{k\to\infty}\frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} - \lim_{k\to 0}\frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} = \frac{\sigma}{\sqrt{1+\sigma^2}} + 1$$

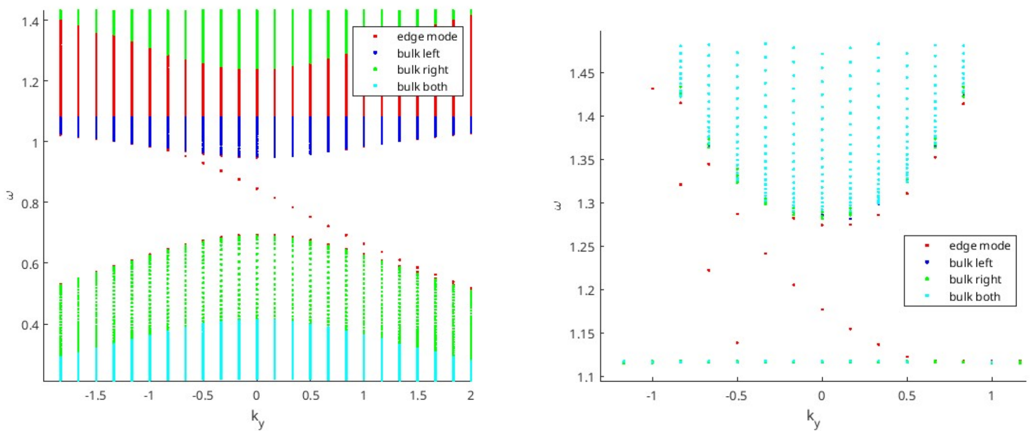

Figure 2: Left: Numerical calculation of eigenvalues while varying ω_p from 0.26 to 0.78 around the critical value $\omega_- = 0.52$ for parameter values $\Omega = 1$ and $k_z = 0.75$. One topologically protected edge state is seen in agreement with Chern number prediction and previous literature. Right: Numerical calculation of eigenvalues varying Ω from -1 to 1 with parameter values $k_z = 0, \omega_p = 1$ without Transverse Electric mode. Two topologically protected edge states are predicted in the band gap between Transverse Magnetic modes, in agreement with previous literature.

5 Comparison with Numerical Results

Considering parameter values which vary in x, we can numerically calculate the eigenvalues of (1) around a boundary between critical values at which band crossings occur by approximating ∂_x by a finite difference matrix and calculating eigenvectors numerically.

Figure 2 shows calculations for two instances where topologically protected edge states are predicted numerically. The first frame shows numerical evidence of the TLCW and confirms the results of [2] [12][10]. In addition, we can compare these results with the difference in analytically calculated Chern numbers from the previous section. The TLCW is the edge state between the 1st and 2nd band at the edge between regions I and II. We see in fact that across this boundary that $\Delta C_1 = -1$ and $\Delta C_2 = 1$. Therefore, even though C_2 is not integer-valued, the number of edge modes still corresponds to the difference of Chern numbers.

The right frame shows the band gap between Transverse Magnetic modes with $k_z = 0$ at the edge between $\Omega = \pm 1$. Here the difference in Chern numbers across the gap is 2 for the upper TM mode, but $2\sigma/\sqrt{1+\sigma^2}$ for the lower mode. Clearly this non-integer difference is not consistent with a Bulk-Edge Correspondence, but as shown in [10] regularization of (1) in a physically consistent way produces integer Chern numbers and may allow for a BEC in all cases.

Appendix A: $\Omega = 0$ Calculations

In this case from (1) with $\Omega = 0$:

$$\begin{aligned} -i\omega_p E &= \omega v \\ i\omega_p v - k \times B &= \omega E \\ k \times E &= \omega B \end{aligned} \tag{19}$$

we can now eliminate E easily:

$$\begin{aligned} E &= i\frac{\omega}{\omega_p} v \\ i\omega_p(1 - \frac{\omega^2}{\omega_p^2})v &= k \times B \\ i\frac{\omega}{\omega_p} k \times v &= \omega B \end{aligned} \tag{20}$$

First consider $\omega = 0$. From (20) we immediately get that $E = 0$, $i\omega_p v = k \times B$, and the third equation becomes trivial. For arbitrary $B \in \mathbb{C}^3$ we can plug these values back into (19), confirming that $-iE = 0 = \omega v$, $k \times E = 0 = \omega B$ and:

$$i\omega_p v - k \times B = i\omega_p(\frac{-i}{\omega_p} k \times B) - k \times B = 0$$

Therefore $\omega = 0$ is an eigenvalue with eigenvector:

$$\omega_0 = 0$$

$$\Psi_0 = (-\frac{i}{\omega_p} k \times B, 0, B)$$

for $B \in \{e_1, e_2, e_3\}$ or any other orthonormal basis of $\mathbb{C}^3$. We have seen that for any parameter values $\Psi = (0, 0, k)$ is a zero eigenvector so it is more natural to choose a basis which includes k and two vectors perpendicular to k. However, for arbitrary k there is no natural way to choose these two vectors. Choosing a basis based on k will prove to be essential later.

Next consider the Ansatz $\omega = \omega_p$. Plugging this into (20) gives the eigenvalue/eigenvector pair:

$$\omega_1 = \omega_p$$

$$\Psi_1 = (k, ik, 0)$$

Modes with $\omega = \omega_p$ are often called the plasma or Langmuir oscillations.

Finally, consider the case that $v \perp k$. Then from (20) we have that:

$$\omega B = i\frac{\omega}{\omega_p}(k \times v) \Rightarrow k \times B = \frac{i}{\omega_p} k \times (k \times v) = -\frac{i}{\omega_p} k^2 v$$

and

$$i\omega_p(1 - \frac{\omega^2}{\omega_p^2})v = -\frac{i}{\omega_p} k^2 v \Rightarrow$$

$$(\omega^2 - \omega_p^2)v = k^2 v \Rightarrow$$
$$\omega^2 = k^2 + \omega_p^2$$

Plugging this value into (19) gives the eigenvalue/eigenvector pair:

$$\omega_2 = \sqrt{k^2 + \omega_p^2}$$

$$\Psi_2 = (\omega_p v, i\omega_2 v, ik \times v)$$

We must choose $v \perp k$, but this subspace has dimension 2, so this eigenvector has a multiplicity of 2. To summarize we have found the eigenvalues and eigenvectors:

$$\begin{aligned}
\omega_0 &= 0 \\
\Psi_0 &= (-ik \times B, 0, \omega_p B)(3) \\
\omega_1 &= \omega_p \\
\Psi_1 &= (k, ik, 0) \\
\omega_2 &= \sqrt{k^2 + \omega_p^2} \\
\Psi_2 &= (\omega_p v, i\omega_2 v, ik \times v)(2)
\end{aligned} \tag{21}$$

Here we have 3 positive eigenvalues, so by the $\pm$ symmetry shown in Section 4.1 we can obtain 3 corresponding negative eigenvalues and their corresponding eigenvectors. Combined with 3 zero eigenvectors we have found all the eigenvectors of the system for $\Omega = 0$.

Now apply these eigenvectors to calculate Chern numbers. Since $B_0 = 0$ the choice of coordinate basis is arbitrary. Therefore WLOG assume that $k_z \neq 0$. This allows us to denote an orthogonal eigenbasis which is smooth in $k_\perp$ as follows:

$$\omega_0 = 0,\ \omega_1 = \omega_p,\ \omega_2 = \sqrt{k^2 + \omega_p^2}$$

$$\Psi_0^{(1)} = (-ik \times \hat{x}, 0, \omega_p \hat{z})$$

$$\Psi_0^{(2)} = (-ik \times \hat{y}, 0, \omega_p \hat{y})$$

$$\Psi_0^{(3)} = (-ik \times \hat{z}, 0, \omega_p \hat{z})$$

$$\Psi_1 = (k, ik, 0)$$

$$\Psi_2^{(1)} = (k \times \hat{x},\ i\omega_2(k \times \hat{x}),\ ik \times (k \times \hat{x}))$$

$$\Psi_2^{(2)} = (\omega_p k \times \hat{y},\ i\omega_2(\omega_p k \times \hat{y}),\ ik \times (k \times \hat{y}))$$

From (13) and (15) it's easy to see that:

$$F_1 = -2\,\mathrm{Im}\left((\partial_{k_x}\Psi_1)^* \partial_{k_y}\Psi_1\right) = (\hat{x}, -i\hat{x}, 0)\begin{pmatrix}\hat{y} \\ i\hat{y} \\ 0\end{pmatrix} = 0 \ \ \forall k$$

$$\Rightarrow C_1 = 0$$

It's also straightforward to calculate:

$\partial_{k_x}(k \times \hat{x}) = 0 \quad \partial_{k_x}(k \times \hat{y}) = \hat{z} \quad \partial_{k_x}(k \times \hat{z}) = -\hat{y}$

$\partial_{k_y}(k \times \hat{x}) = -\hat{z} \quad \partial_{k_y}(k \times \hat{y}) = 0 \quad \partial_{k_y}(k \times \hat{z}) = \hat{x}$

and so

$\partial_{k_x}\Psi_0^{(1)} = (0,0,0) \quad \partial_{k_x}\Psi_0^{(2)} = (-i\hat{z},0,0) \quad \partial_{k_x}\Psi_0^{(3)} = (i\hat{y},0,0)$

$\partial_{k_y}\Psi_0^{(1)} = (i\hat{z},0,0) \quad \partial_{k_y}\Psi_0^{(2)} = (0,0,0) \quad \partial_{k_y}\Psi_0^{(3)} = (-i\hat{x},0,0)$

Therefore again we have:

$$(\partial_{k_x}\Psi_0)^*\Psi_0 = 0 \Rightarrow F_0 = 0 \;\; \forall k_\perp \Rightarrow C_0 = 0$$

Finally, for Ψ_2 we can calculate:

$$\partial_{k_x}\Psi_2^{(1)} = \Big(0, \quad 0, \quad 0, \quad 0, \quad i\tfrac{2|k|k_x}{\omega_2}k_z, \quad -i\tfrac{2|k|k_x}{\omega_2}k_y, \quad 0, \quad ik_y, \quad ik_z\Big)^T$$

$$\partial_{k_x}\Psi_2^{(1)} = \Big(0, \quad 0, \quad -1, \quad i\tfrac{2|k|k_y}{\omega_2}k_z, \quad -i\tfrac{2|k|k_y}{\omega_2}k_y, \quad -\omega_2, \quad -2ik_y, \quad ik_x, \quad 0\Big)^T$$

Although this time $(\partial_{k_x}\Psi_2)^*\partial_{k_y}\Psi_2 \neq 0$ we can see that each component of $\partial_{k_x}\Psi_2$ and the corresponding component of $\partial_{k_y}\Psi_2$ are either purely imaginary or purely real. Therefore $\mathrm{Im}\left[(\partial_{k_x}\Psi_2)^*\partial_{k_y}\Psi_2\right] = 0 \Rightarrow F_2 = 0 \;\; \forall k_\perp$ so again $C_2 = 0$. Rotational symmetry gives the same result for $\Psi_2^{(2)}$ as $\Psi_2^{(1)}$. Therefore if $B_0 = 0$ we have that all bands are topologically trivial.

Appendix B: $k_z = 0$ Calculations

$$\begin{aligned} -i\Omega\hat{z} \times v - i\omega_p E &= \omega v \\ i\omega_p v - k \times B &= \omega E \\ k \times E &= \omega B \end{aligned} \tag{22}$$

noting in this case $k \perp \hat{z}$. Consider first and zero eigenvectors with $v \parallel \hat{z}$. This gives:

$$-i\Omega\hat{z} \times v - i\omega_p E = 0 \Rightarrow E = 0$$

$$i\omega_p v - k \times B = 0 \Rightarrow k \times B = i\omega_p \hat{z}$$

Assuming that $B \perp k$ we can solve for B:

$$k \times (k \times B) = -k^2 B = i\omega_p k \times \hat{z} \Rightarrow$$

$$B = -\frac{i\omega_p}{k^2} k \times \hat{z}$$

which gives another 0 eigenvector:

$$\Psi_0 = (k^2\hat{z}, 0, -i\omega_p k \times \hat{z})$$

Now consider a zero eigenvector with $E \parallel k$. Clearly we have $k \times E = 0$ and plugging into the first equation gives:

$$i\omega_p E = i\omega_p k = -i\Omega\hat{z} \times v$$

If we assume that both E and v are perpendicular to $\hat{z}$ then we get:

$$-\Omega\hat{z} \times (\hat{z} \times v) = \Omega v = \omega_p \hat{z} \times k \Rightarrow$$

$$v = \frac{1}{\sigma}\hat{z} \times k$$

Plugging into the second line gives:

$$i\omega_p v = \frac{i\omega_p}{\sigma}\hat{z} \times k = -\frac{i\omega_p}{\sigma} k \times \hat{z} = -k \times B \Rightarrow$$

$$B = -\frac{i\omega_p}{\sigma}\hat{z}$$

Therefore the $k_z = 0$ case produces a third zero eigenvector:

$$\Psi_0^{(3)} = (k \times \hat{z}, -\sigma k, i\omega_p \hat{z})$$

Now recall that assuming E, v are elliptically polarized in the x/y plane led to (8), and the $k_z = 0$ case corresponds to $\theta = \pi/2$, which yields:

$$\begin{bmatrix} S & iD & 0 \\ -iD & S - n^2 & 0 \\ 0 & 0 & P - n^2 \end{bmatrix} E = 0$$

The first non-trivial solution to this equation is

$$n^2 = P \Rightarrow k^2 = \omega^2 - \omega_p^2 \Rightarrow$$

$$\omega_2 = \sqrt{k^2 + \omega_p^2}$$

Clearly the E associated with this solution is $E = \hat{z}$. Plugging this in to (1) we get:

$$k \times E = \omega_2 B \Rightarrow$$

$$B = \frac{1}{\omega_2} k \times \hat{z}$$

$$i\omega_p v - k \times B = \omega_2 E \Rightarrow$$

$$v = \frac{i\hat{z}}{\omega_p \omega_2}(k^2 - \omega_2^2) = -\frac{i\hat{z}\omega_p}{\omega_2}$$

Summarizing, the eigenvector/eigenvalue pair is:

$$\omega_2 = \sqrt{k^2 + \omega_p^2}$$

$$\Psi_2 = (-i\omega_p \hat{z}, \omega_2 \hat{z}, k \times \hat{z})$$

Due to the fact that $E \perp k$ this wave is often referred to as the Transverse Electric wave [7].

The final eigenvalues solve the equation:

$$S(S - n^2) - D^2 = 0 \Rightarrow$$

$$n^2 = \frac{S^2 - D^2}{S} = \frac{RL}{S}$$

Plugging in the definitions of n, R, L, S gives:

$$\left(\frac{k}{\omega}\right)^2 = \frac{\left(1 - \frac{\omega_p^2}{\omega(\omega+\Omega)}\right)\left(1 - \frac{\omega_p^2}{\omega(\omega-\Omega)}\right)}{\frac{1}{2}\left(2 - \left(\frac{\omega_p^2(\omega+\Omega)+\omega_p^2(\omega-\Omega)}{\omega(\omega^2-\Omega^2)}\right)\right)} \Rightarrow$$

$$k^2 = \frac{\omega^2(\omega^2 - \Omega^2) - 2\omega_p^2\omega^2 + \omega_p^4}{\omega^2 - \Omega^2 - \omega_p^2}$$

Denoting the upper-harmonic frequency $\omega_{uh}^2 = \Omega^2 + \omega_p^2$ we get:

$$k^2 = \frac{\omega^2(\omega^2 - \Omega^2) - 2\omega_p^2\omega^2 + \omega_p^4}{\omega^2 - \omega_{uh}^2}$$

Immediately we see that there as $\omega \to \pm\omega_{uh}$, $k^2 \to \infty$. With some algebraic manipulation we get a quartic equation for the remaining eigenvalues:

$$\omega^4 - \omega^2(k^2 + \Omega^2 + 2\omega_p^2) + (\omega_p^4 + k^2\omega_{uh}^2) = 0$$

Absence of odd-degree terms means we can solve using the quadratic formula. Using the useful substitution $\omega_h^2 = \Omega^2 + 2\omega_p^2$ we get:

$$\begin{aligned}\omega^2 &= \frac{1}{2}\left((k^2 + \omega_h^2) \pm \sqrt{(k^2 + \omega_h^2)^2 - 4(\omega_p^4 + k^2\omega_{uh}^2)}\right) \\ &= \frac{1}{2}\left((k^2 + \omega_h^2) \pm \sqrt{k^4 + 2(2\omega_p^2 - k^2)\Omega^2 + \Omega^4}\right)\end{aligned}$$

This gives us 2 positive and 2 negative bands. We can also see that if $n^2 \neq P$, then we must have $E_z = 0$, so in this case E is polarized in the x/y plane. Since $k \times E = \omega B$, then we see that $B \parallel \hat{z}$. This is why these last two modes are sometimes denoted the Transverse Magnetic waves [7]. From (2) we know that $k \times B = \omega\epsilon E$. Therefore in this case we have:

$$B = -\hat{z}$$

$$E = \epsilon^{-1}\frac{k}{\omega} \times \hat{z}$$

$$v = \frac{i}{\omega_p}(I - \epsilon^{-1})k \times \hat{z}$$

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