# Topological Properties of Cold Plasma

## M.S. Thesis

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**Abstract**

In this paper we analytically calculate the Chern numbers of the cold plasma light-matter interacting system and analyze the validity of the bulk-edge correspondence (BEC) with respect to these invariants. It has been a well-studied fact that regularization of the problem is needed in this system in order to restore integer valued, or even invariant, Chern numbers [1]. Through systematic calculation of Chern numbers and comparison to numerical simulations of edge states we find that previous regularization techniques, while successful in predicting some edge modes, fail to validate the BEC in other cases. In light of this, we propose a new regularization method and show that this method is consistent with the BEC in all cases through use of numerical simulations of edge states.

# Contents

# 1 Introduction

The focus of this paper is on cold plasma, which is characterized by a low enough temperature that the motion of ions other than electrons is neglected- essentially an electron gas. This approximation has various applications in low energy plasma physics [2, 3, 4] and photonics [1, 5, 6, 7, 8]. The wave dynamics for cold plasma and plasmas in general has been studied for decades [9], but recent advances in topological physics suggest that topological arguments may be able to predict novel behaviors in plasmas. For a review of the topological physics in general, topological invariants, and topological physics in plasma see [10, 11, 12].

Recently, one topologically protected edge state at the edge between two regions of topologically distinct electron densities, termed the Topological Langmuir Cyclotron Wave (TLCW) has been studied. Parker et. al. first predicted the mode using numerical methods in [2], and Fu and Qin provided a detailed analysis of this mode in [13, 14] and characterized the topological phases of the system in [15].

The preferred topological invariants used in this system and other hydro-dynamic like systems are Chern numbers. The main goal of this paper is to expand on the results of [15, 13, 2] by analytically calculating all the Chern numbers of the system, which were previously only calculated numerically. It is apparent from the fact that some Chern numbers are not integer-valued that the Bulk-Edge Correspondence (BEC) does not exist for this system without some modification. Although we do not attempt to prove the existence or non-existence of the BEC here, some intuition regarding its validity is analyzed in Section 5 by comparing with some numerical results, particularly in the case where some regularization factors are added in.

Some analytic calculations of Chern numbers have been done for particular parameter values ($k_z = 0$) [16, 6] which are of particular interest in photonics. In this case we shall see that the system as we have defined it actually breaks into two de-coupled systems and fundamentally alters the topological structure.

We will begin by deriving the Hamiltonian of the system and analyzing its symmetry properties. Then, all the eigenvectors needed to calculate Chern numbers are derived. Although analytical expressions are not available for eigenvectors in the general case, we will find that only certain limits of eigenvectors are needed to calculate the Chern numbers of this system [1]. Applying these results allows us to calculate some Chern numbers directly and infer all others from symmetries of the system. Finally, the $k_z = 0$ case is analyzed independently and compared to the general case, and some numerical results are presented to develop some intuition on whether a BEC can exist in general.

## 1.1 Derivation of Equations

The following derivation largely mirrors the derivation in [9]. We are interested here in waves in a plasma biased by a constant incident magnetic field $B_i = \hat{z}B_0$. We start with Maxwell's equations and Lorentz force equation for a cloud of electrons with density $n_e$, velocity $v$, and charge $q_e$. In order to linearize the problem, only the incident field is considered in Lorentz's equation, which is merely to assume that the magnitude of any waves present in the plasma are small compared to the incident field.

$$m_e \frac{\partial v}{\partial t} = q_e(E + v \times B_i)$$

$$c^2 \nabla \times B = \frac{1}{\epsilon_0} J + \frac{\partial E}{\partial t} = \frac{n_e q_e v}{\epsilon_0} + \frac{\partial E}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Note that we treat the electron density as a constant, or slowly varying with respect to frequency, mean density about which the electrons essentially "vibrate". This assumption, which amounts to assuming that to zero-th order electron velocity $v^{(0)} = 0$ is the essential simplification of the cold plasma model, and allows us to assume that the Lorentz force only depends on the incident magnetic $B_i$ and not $B$. For a detailed discussion of the validity of the cold-plasma approximation and derivation from kinetic hot-plasma equations see Stix Chapters 10.7 and 11.5 [9]. SI units are used above, but we will re-normalize the system so that $v$, $E$, and $k$ all have units of electric field $\frac{m \cdot kg}{s^2 \cdot C} = N/C$. Now we will make the substitutions:

$$v_n = v \frac{m_e \omega_p}{q_e} \quad B_n = cB$$

We define the electron plasma frequency and electron gyro-frequency respectively as:

$$\omega_p = \sqrt{\frac{n_e q_e^2}{m_e \epsilon_0}} \quad \Omega = \frac{B_0 q_e}{m_e}$$

This gives the coupled equations:

$$\partial_t v_n = \omega_p E - \Omega \hat{z} \times v_n$$

$$\partial_t E = c \nabla \times B_n - \omega_p v_n$$

$$\partial_t B_n = -c \nabla \times E$$

which defines the 1st order PDE:

$$\partial_t \psi = H\psi, \quad H = \begin{bmatrix} -\Omega \hat{z} \times & \omega_p & 0 \\ -\omega_p & 0 & c\nabla \times \\ 0 & -c\nabla \times & 0 \end{bmatrix}, \quad \psi = \begin{pmatrix} v_n \\ E \\ B_n \end{pmatrix} \tag{1}$$

Taking the Fourier transform in time and space and normalizing the spatial dual variables $\mathbf{k} = c(k_x, k_y, k_z)^T$ we get the eigenvalue problem:

$$\omega \tilde{\psi} = H(\mathbf{k}, \Omega, \omega_p)\tilde{\psi}, \quad H(\mathbf{k}, \Omega, \omega_p) = \begin{bmatrix} i\Omega \hat{z} \times & -i\omega_p & 0 \\ i\omega_p & 0 & -\mathbf{k} \times \\ 0 & \mathbf{k} \times & 0 \end{bmatrix} \tag{2}$$

For the rest of this paper we will drop the subscripts and ˜ notation and simply regard $v, E, B$ as the normalized Fourier transform of themselves as defined above. Here and throughout we use the shorthand for the cross product matrix:

$$u\times = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

for $u \in \mathbb{C}^3$. Because electron motion in the $\hat{z}$ direction is unaffected by the incident magnetic field, $k_z$ can be treated as a parameter of the system along with $\Omega$- proportional to $B_0$- and $\omega_p$- proportional to $\sqrt{n_e}$.

## 1.2 Susceptibility Tensor

In many cases it will be useful to eliminate $v$ from the system using the susceptibility tensor $\chi$ such that $v = \chi E$. Writing out the $v$ row of $H$ gives:

$$i\Omega\hat{z} \times v - i\omega_p E = \omega v$$

Define the circularly left and right polarized vectors:

$$v_\pm = e_\pm = \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}$$

Then, after multiplying by $-i$ and rearranging we get:

$$\Omega\hat{z} \times v_\pm + i\omega v_\pm = \omega_p E_\pm \Rightarrow$$

$$\Omega \begin{pmatrix} \mp i \\ 1 \\ 0 \end{pmatrix} + i\omega \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} = \begin{pmatrix} i(\omega \mp \Omega) \\ \Omega \mp \omega \\ 0 \end{pmatrix} = \begin{bmatrix} i\omega & -\Omega & 0 \\ \Omega & i\omega & 0 \\ 0 & 0 & 0 \end{bmatrix} v_\pm = \omega_p E_\pm$$

Similarly if we assume that $v = \hat{z}$ then

$$\Omega\hat{z} \times v + i\omega v = i\omega v = \omega_p E \Rightarrow$$

$$E_z = i\frac{\omega}{\omega_p} v_z$$

Therefore we have that:

$$\frac{1}{\omega_p} \begin{bmatrix} i\omega & -\Omega & 0 \\ \Omega & i\omega & 0 \\ 0 & 0 & i\omega \end{bmatrix} v = E$$

since $\{\hat{e}_+, \hat{e}_-, \hat{z}\}$ form an orthonormal basis of $\mathbb{R}^3$. For finite $\omega_p$, $\Omega$, and $\omega$ this matrix is invertible as long as $\omega \neq \Omega$. Therefore assuming that $\omega \neq \Omega$ we get:

$$\chi = \left( \frac{1}{\omega_p} \begin{bmatrix} i\omega & -\Omega & 0 \\ \Omega & i\omega & 0 \\ 0 & 0 & i\omega \end{bmatrix} \right)^{-1} = \omega_p \begin{bmatrix} \frac{i\omega}{\Omega^2-\omega^2} & \frac{\Omega}{\Omega^2-\omega^2} & 0 \\ -\frac{\Omega}{\Omega^2-\omega^2} & \frac{i\omega}{\Omega^2-\omega^2} & 0 \\ 0 & 0 & -i\frac{1}{\omega} \end{bmatrix}$$

If we substitute this into (2) the first line is eliminated (we used it to derive $\chi$) and the Hamiltonian becomes:

$$H = \begin{bmatrix} i\omega_p\chi & -\mathbf{k}\times \\ \mathbf{k}\times & 0 \end{bmatrix} \tag{3}$$

$$H\begin{bmatrix} E \\ B \end{bmatrix} = \omega \begin{bmatrix} E \\ B \end{bmatrix}$$

# 2 Eigenvalue Symmetry

The symmetries of the system were noted in [15], which we make explicit using unitary and anti-unitary transformations below. The $k_x/k_y$ plane rotational symmetry is particularly important in obtaining tractable calculations for Chern numbers.

## 2.1 $\pm\mathbf{k}$ Symmetry

Define the projection:

$$\Gamma_k = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & -I_3 \end{bmatrix}$$

Then we have that $H(\mathbf{k})$ is unitarily equivalent to $H(\mathbf{k})$ through:

$$H(-\mathbf{k}) = \Gamma_k H(\mathbf{k})\Gamma_k$$

Assume now that $\omega, x = (v, E, B)^T$ is an eigenvalue/eigenvector pair of $H(\mathbf{k})$. Then we can conclude that:

$$H(-\mathbf{k})\Gamma_k x = \Gamma_k H(\mathbf{k})x = \omega(\Gamma_k x) \Rightarrow$$
$$\omega(\Gamma_k x) = H(-\mathbf{k})(\Gamma_k x)$$

so we have that $\omega$ is an eigenvalue of $H(-\mathbf{k})$ with eigenvector:

$$\Gamma_k x = \begin{bmatrix} v \\ E \\ -B \end{bmatrix}$$

## 2.2 $\pm\omega$ Symmetry

Now notice that, defining $K$ as the operator of element-wise complex conjugation, or $Kx = \bar{x}$:

$$\Gamma_k K H(\mathbf{k}) K \Gamma_k = \Gamma_k \bar{H}(\mathbf{k})\Gamma_k = -H(\mathbf{k})$$

From which follows that if $\omega, x$ is an eigenvalue/eigenvector pair of $H(\mathbf{k})$ then:

$$H(\mathbf{k})(K\Gamma_k x) = -K\Gamma_k(H(\mathbf{k})x) = -\omega(K\Gamma_k x)$$

Therefore if $\omega \neq 0$ is an eigenvalue of $H(\mathbf{k})$ then so is $-\omega$. By symmetry then we can deduce that there are (at most) four positive eigenvalues $(\omega_1, \omega_2, \omega_3, \omega_4)$ and four negative eigenvalues $\omega_{-n} = -\omega_n$, $n = \{1, 2, 3, 4\}$. In addition if $x = (v, E, B)^T$ is an eigenvector of $\omega_n$ then $K\Gamma_k x = (\bar{v}, \bar{E}, -\bar{B})^T$ is an eigenvector corresponding to $\omega_{-n}$.

## 2.3 $\pm\Omega$ Symmetry

Now consider the projection:

$$\Gamma_\Omega = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & -I_3 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

We can verify from (2) that:

$$-\Gamma_\Omega H(\Omega)\Gamma_\Omega = H(-\Omega)$$

Assuming again that $\omega_n, x$ is an eigenvalue/eigenvector pair of $H$ this gives:

$$-\Gamma_\Omega H(\Omega)x = -\Gamma_\Omega \omega_n x = H(-\Omega)\Gamma_\Omega x \Rightarrow$$

$$H(-\Omega)(\Gamma_\Omega x) = -\omega_n(\Gamma_\Omega x)$$

Therefore $\omega_{-n} = -\omega_n$ is an eigenvalue of $H(-\Omega)$ with eigenvector:

$$\Gamma_\Omega x = \begin{bmatrix} v \\ -E \\ B \end{bmatrix}$$

Combining this the result from the previous section we get that $\omega_n$ is also an eigenvalue of $H(-\Omega)$ with eigenvector:

$$K\Gamma_k\Gamma_\Omega x = \begin{bmatrix} \bar{v} \\ -\bar{E} \\ -\bar{B} \end{bmatrix}$$

## 2.4 $k_x/k_y$ Plane Rotational Symmetry

Consider rotating the $\mathbf{k}$ vector in the $x/y$ plane. Suppose that after $\mathbf{k}$ is rotated in the $x/y$ plane by and angle $\theta$, the new Hamiltonian is $H_\theta$. Using the usual 2-d rotation matrix we can see that if we rotate $\mathbf{k}$ by an angle $\theta$ in the $x/y$ plane we get:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} k_x\cos\theta - k_y\sin\theta \\ k_x\sin\theta + k_y\cos\theta \\ k_z \end{bmatrix} = \begin{bmatrix} k_{x2} \\ k_{y2} \\ k_z \end{bmatrix} = \mathbf{k}(\theta)$$

Denote the two matrices:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R(-\pi/2).$$

One can easily show that $R$ and $S$ are orthonormal. Remembering the definition of $\mathbf{k}\times$:

$$\mathbf{k}\times = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

we can see that we want to rotate the vector $\begin{bmatrix} k_y & -k_x & 0 \end{bmatrix}^T$ in the third column and the same corresponding row vector in the third row.

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = S\mathbf{k} = \begin{bmatrix} k_y \\ -k_x \\ k_z \end{bmatrix}$$

Noticing that $S^{-1} = S^T$ we can see that $x/y$ plane rotation by $\theta$ can be accomplished on the $(k_y, -k_x)^T$ vector by the transformation:

$$S^T RS \begin{bmatrix} k_y \\ k_x \\ 0 \end{bmatrix} = \begin{bmatrix} k_{y2} \\ -k_{x2} \\ 0 \end{bmatrix}$$

However, straightforward computation shows that $S^T RS = R$. Rotating the last row would then be equivalent to the operation:

$$\left( R \begin{bmatrix} k_y \\ -k_x \\ 0 \end{bmatrix} \right)^T = \begin{bmatrix} k_y & -k_x & 0 \end{bmatrix} R^T$$

Noticing that the upper left portion of $\mathbf{k}\times$ is just a multiple of $S$ and therefore commutes with $R$ we can see that:

$$R[\mathbf{k}\times]R^T = \begin{bmatrix} 0 & -k_z & k_{y2} \\ k_z & 0 & -k_{x2} \\ -k_{y2} & k_{x2} & 0 \end{bmatrix} = [\mathbf{k}(\theta)\times]$$

which can also be verified by straightforward calculation. Therefore denoting:

$$\mathbf{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix},$$

noticing that $\hat{z}\times = -S$ and using the fact that $R$ is orthonormal we get:

$$H_\theta = \mathbf{R}H\mathbf{R}^T$$

This derivation was shown to highlight the symmetries of the problem but straightforward computation also verifies this fact. Assuming again that $x$ is and eigenvector of $H$ with eigenvalue $\omega$ we get:

$$H_\theta \mathbf{R} = \mathbf{R}H \Rightarrow$$

$$H_\theta(\mathbf{R}x) = \mathbf{R}Hx = \omega(\mathbf{R}x)$$

Therefore $\omega$ is also an eigenvalue of $H_\theta$ with eigenvector $\mathbf{R}x$, which simply rotates the $x$ and $y$ components of $v, E, B$ respectively by $\theta$.

## 3 Eigenvector Calculations

As shown above, the eigenvalues have well-defined $\pm$ symmetry, so we will only show derivations for the positive and zero-valued eigenvalues. First note that there is a zero eigenvalue for any parameter choice which is $\Psi = (0, 0, \mathbf{k})$. Plugging this Ansatz into (2) we get:

$$\mathbf{k} \times E = \mathbf{k} \times 0 = 0$$

$$i\omega_p v - \mathbf{k} \times B = -\mathbf{k} \times \mathbf{k} = 0$$
$$-i\Omega\hat{z} \times v - i\omega_p E = 0$$

regardless of the values of $k_z, k_\perp, \omega_p, \Omega$. Therefore we have the universal eigenvalue/eigenvector pair:

$$\omega_0 = 0$$
$$\Psi_0 = (0, 0, \mathbf{k})^T$$

Unfortunately this is the only readily available eigenvalue/eigenvector pair which applies to all parameter values. Below we consider some instructive and especially useful cases where eigenvalues can be analytically derived.

By the $\pm\omega$ symmetry described above we need only consider positive eigenvalues $(\omega_1, \omega_2, \omega_3, \omega_4)$ and their corresponding eigenvectors $\Psi_1, \Psi_2, \Psi_3, \Psi_4$. As discussed in Section 1 we treat $k_z$ as a parameter of the system and therefore it is natural to denote $\mathbf{k} = (\mathbf{k}_\perp, k_z)^T$, $\mathbf{k}_\perp = (k_x, k_y)$, and in addition $|\mathbf{k}| = k$, $|\mathbf{k}_\perp| = k_\perp$. In order to calculate topological invariants we will need the limiting eigenvectors $\Psi_n(k_\perp = 0)$ and $\Psi_n(k_\perp \to \infty)$ (see Section 4). We will begin by calculating these eigenvectors for strictly positive values of parameters $(\Omega, \omega_p, k_z)$ noting that we may find expressions for negative $\Omega, k_z$ by applying the unitary transformations described in Sections 2.1 and 2.3. $\omega_p$, being proportional to $\sqrt{n_e}$, is treated as inherently positive while the $\Omega = 0$ and $k_z = 0$ cases are singular and treated separately.

### 3.1 Case: $k_\perp = 0$

The $k_\perp = 0$ case is important for two reasons. First, as shown by Fu and Qin by extensive numerical simulations [15], band crossings only happen when $k_\perp = 0$, $\Omega = 0$, or $k_z = 0$, so these parameter values are important for considering topological phase transitions. Second, as we will see later, the $k_\perp = 0$ eigenvectors are essential for calculating Chern numbers. Calculations are presented below along with analysis of band crossings that will be especially useful in calculating topological invariants. We use the conventions and many of the same analytical tools as [9] to derive dispersion relations and eigenvectors of the system for this case and the $k_z = 0$ case.

Setting $k_\perp = 0$ gives us the eigenvalue equation from (2):

$$\begin{aligned} i\Omega\hat{z} \times v - i\omega_p E &= \omega v \\ i\omega_p v - k_z\hat{z} \times B &= \omega E \\ k_z\hat{z} \times E &= \omega B \end{aligned} \tag{4}$$

We will consider the non-trivial case $k_z \neq 0$. Consider the Ansatz for a plasma oscillation, or $\omega_1 = \omega_p$. From the first line of (4) we get:

$$-i\Omega \hat{z} \times v - i\omega_p E = \omega_p v$$

If we consider the case that $v$ is either real or purely imaginary it's apparent from this equation that $i\Omega \hat{z} \times v \perp \omega_p v$. Therefore a non-trivial solution is $E = \hat{z}$ and $v = -i\hat{z}$. It follows from the rest of (4) that:

$$\omega_p B = k_z \hat{z} \times E = k_z \hat{z} \times \hat{z} = 0 \Rightarrow B = 0$$

$$i\omega_p v - k_z \hat{z} \times B = \omega_p \hat{z} = \omega_p E$$

Therefore we have the plasma eigenmode:

$$\omega_1 = \omega_p$$

$$\Psi_1 = \frac{1}{\sqrt{2}}(\hat{z}, i\hat{z}, 0)$$

For the remaining 3 positive eigenvalues we will utilize the susceptibility tensor calculated in Section 1.1 $v = \chi E$. As we showed that the vectors right and left circularly polarized vectors in the x/y plane $E_\pm = v_\pm = (1, \pm i, 0)$ are eigenvectors of $\chi$ these may be good Ansatz's for eigenvectors of the whole system.

With some rearranging of (3) we get:

$$k \times E = \omega B \Rightarrow k \times B = \frac{k}{\omega} \times (k \times E)$$

$$\omega(I - i\frac{\omega_p}{\omega}\chi)E + k \times B = 0 \Rightarrow$$

$$\frac{k}{\omega} \times (\frac{k}{\omega} \times E) + (I - i\frac{\omega_p}{\omega}\chi)E = 0$$

Setting $I - i(\omega_p/\omega)\chi = \epsilon$ and $n = k/\omega$ this gives what is commonly known as the homogeneous plasma wave equation:

$$n \times (n \times E) + \epsilon E = 0 \tag{5}$$

The dielectric tensor $\epsilon$ is commonly written as:

$$\epsilon = \begin{bmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

where one can check easily that:

$$R = 1 - \frac{\omega_p^2}{\omega(\omega + \Omega)} \quad L = 1 - \frac{\omega_p^2}{\omega(\omega - \Omega)}$$

$$S = \frac{1}{2}(R + L) \quad D = \frac{1}{2}(R - L) \quad P = 1 - \frac{\omega_p^2}{\omega^2}$$

Assuming WLOG that $k_y = n_y = 0$ due to rotational invariance and denoting $\theta$ as the angle between $n$ and $\hat{z}$ we can represent the operator $[n \times (n\times]$ in matrix form and write the plasma wave equation as:

$$\begin{bmatrix} S - n^2\cos^2\theta & iD & n^2\cos\theta\sin\theta \\ -iD & S - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P - n^2\sin^2\theta \end{bmatrix} E = 0 \tag{6}$$

This is in fact the dispersion relation for the whole system, however it is not solvable in closed form for arbitrary $\theta$. For the current case $k = k_z\hat{z} \Rightarrow \theta = 0$ and the dispersion relation gives:

$$\begin{bmatrix} S - n^2 & iD & 0 \\ -iD & S - n^2 & 0 \\ 0 & 0 & P \end{bmatrix} E = 0$$

Note that the $P = 1 - \frac{\omega_p^2}{\omega^2} = 0$, $E = \hat{z}$ case has already been addressed. Therefore, as mentioned above we will guess that the remaining eigenmodes have $E$ component polarized in the x/y plane. Assuming that $E_z = 0$ and taking the determinant of the upper left matrix to determine when it is singular gives the equation:

$$(S - n^2)^2 - D^2 = 0$$

Plugging in the definition of $S$ and $D$ gives the dispersion relation:

$$\frac{1}{4}(R+L)^2 + (n^2)^2 - n^2(L+R) - \frac{1}{4}(L-R)^2 = LR + n^2(n^2 - L - R) = 0$$

We can see that solutions to this equation are:

$$n^2 = R \Rightarrow$$

$$\left(\frac{k}{\omega}\right)^2 = 1 - \frac{\omega_p^2}{\omega(\omega+\Omega)} \Rightarrow$$

$$k_z^2 = \omega_R^2 - \frac{\omega_p^2}{1 + \frac{\Omega}{\omega_R}} \tag{7}$$

which is usually called the R-wave, and

$$n^2 = L \Rightarrow$$

$$\left(\frac{k_z}{\omega}\right)^2 = 1 - \frac{\omega_p^2}{\omega(\omega-\Omega)} \Rightarrow$$

$$k_z^2 = \omega_L^2 - \frac{\omega_p^2}{1 - \frac{\Omega}{\omega_L}} \tag{8}$$

which is usually called the L-wave. With some elementary manipulation we can see that the eigenvalues are roots of the cubic polynomials:

$$\omega_R^3 + \Omega\omega_R^2 - (k^2 + \omega_p^2)\omega_R - k^2\Omega = 0 \tag{9}$$

for the R-wave and:

$$\omega_L^3 - \Omega\omega_L^2 - (k^2 + \omega_p^2)\omega_L + k^2\Omega = 0 \tag{10}$$

for the L wave. Assuming that all the roots are real these account for the remaining six eigenvalues. Explicit formulas are possible as well, though they are complicated and not very instructive.

Now we will show that there are two positive L-waves $\omega_{L^-} < \omega_{L^+}$ and one positive R-wave, which are always ordered: $\omega_{L^-} < \omega_R < \omega_{L^+}$. Notice first from (8)(7) that as $k_z \to \infty$, $\omega_L$ has two positive solutions, $\omega_L \to \Omega$ from below, and $\omega_L \to \sqrt{k_z^2 + \omega_p^2} \to \infty$. By continuity of eigenvalues then we know that there are two positive branches of $\omega_L$, one of which is $0 < \omega_{L^-} < \Omega$ and one of which is $\Omega < \omega_{L^+}$ since $k_z^2 \to \infty$ if $\omega_L$ approaches $\Omega$, which would violate continuity of $\omega_{L^\pm}$ as a function of $k_z$. Also notice that for $\Omega > 0$ and $\omega_R > 0$:

$$0 < \frac{\omega_p^2}{1 + \frac{\Omega}{\omega_R}} < \omega_p^2 \quad \Rightarrow \quad k_z^2 < \omega_R^2 < \omega_p^2 + k_z^2.$$

Next since we know that $\omega_{L^+} > \Omega$ so that we also have $0 < \Omega/\omega_{L^+} < 1$ so that:

$$\omega_{L^+}^2 = k_z^2 + \frac{\omega_p^2}{1 - \frac{\Omega}{\omega_{L^+}}} > k_z^2 + \omega_p^2$$

so indeed $\omega_{L^+} > \omega_R$ and in addition $\omega_{L^+} > \omega_p$. Finally since $\omega_{L^-} < \Omega \Rightarrow \Omega/\omega_{L^-} > 1$:

$$\omega_{L^-}^2 - k_z^2 = \frac{\omega_p^2}{1 - \frac{\Omega}{\omega_{L^-}}} < 0 \quad \Rightarrow \quad \omega_{L^-}^2 < k_z^2.$$

To summarize we have proved that:

$$\omega_{L^-}^2 < \min\{\Omega^2, k_z^2\} \leq k_z^2 < \omega_R < k_z^2 + \omega_p^2 < \omega_{L^+},$$

so indeed $\omega_{L^-} < \omega_R < \omega_{L^+}$. Therefore $\omega_{L^-}, \omega_R, \omega_{L^+}$ do not cross but we have yet to determine the ordering of the eigenvalue $\omega_p$ with respect to the other three positive eigenvalues.

We established that $\omega_{L^+} > \sqrt{k_z^2 + \omega_p^2} \geq \omega_p$ so $\omega_{L^+}$ and $\omega_p$ do not cross, but in general we may have that $\omega_p = \omega_R$ or $\omega_p = \omega_{L^-}$. Plugging in $\omega_R = \omega_p = \omega_+$ into (7) gives:

$$k_z^2 = \omega_+^2 - \frac{\omega_+^2}{1 + \frac{\Omega}{\omega_+}} \Rightarrow \Omega\omega_+^2 - k_z^2\omega_+ - \Omega k_z^2 = 0 \Rightarrow$$

$$\omega_+ = \frac{k_z^2 + \sqrt{k_z^4 + 4\Omega^2 k_z^2}}{2\Omega}$$

$$\frac{\omega_+}{\Omega} = \frac{1}{2}\left(\sqrt{\left(\frac{k_z}{\Omega}\right)^4 + 4\left(\frac{k_z}{\Omega}\right)^2} + \left(\frac{k_z}{\Omega}\right)^2\right)$$

Similarly if $\omega_p < \Omega$ we can see that $\omega_{L^-}$ crosses $\omega_p$ since as $k_z \to \infty$, $\omega_{L^-} \to \Omega$ from below. Denoting this crossing $\omega_{L^-} = \omega_p = \omega_-$ we get:

$$k_z^2 = \omega_-^2 - \frac{\omega_-^2}{1 - \frac{\Omega}{\omega_-}} \Rightarrow \Omega\omega_-^2 + k_z^2\omega_- - \Omega k_z^2 = 0 \Rightarrow$$

$$\omega_- = \frac{\sqrt{k_z^4 + 4\Omega^2 k_z^2} - k_z^2}{2\Omega} \Rightarrow$$

$$\frac{\omega_-}{\Omega} = \frac{1}{2}\left(\sqrt{\left(\frac{k_z}{\Omega}\right)^4 + 4\left(\frac{k_z}{\Omega}\right)^2} - \left(\frac{k_z}{\Omega}\right)^2\right)$$

Numerical illustration of band crossings and eigenvalue behavior is shown in Figure 1. The $\omega_-$ crossing gives rise to what is termed the Topological Cyclotron Langmuir wave (TCLW), extensively studied in [13], and has a topologically protected edge state as we shall see below.

Particularly important for the calculation of Chern numbers are the eigenvectors associated with these eigenvalues. Assuming that $E_z = 0$ and plugging in $n^2 = R$ we get:

$$S - n^2 = S - R = -D \Rightarrow$$

$$\begin{bmatrix} -D & iD & 0 \\ -iD & -D & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \Rightarrow$$

$$E_R = e_- = (1, -i, 0)^T$$

Similarly if $n^2 = L$ we get $S - n^2 = D$ and $E_L = e_+ = (1, i, 0)^T$, which represent left- and right-circularly polarized electric field as our notation suggested above. We can use the equations $k \times E = \omega B$ and $\chi E = v$ to obtain the full eigenvectors for R- and L-modes:

$$\Psi_R = \left(-i\frac{\omega_p}{\Omega + \omega_R} e_-, e_-, i\frac{k}{\omega_R} e_-\right)$$

$$\Psi_L = \left(i\frac{\omega_p}{\Omega - \omega_L} e_+, e_+, -i\frac{k}{\omega_L} e_+\right)$$

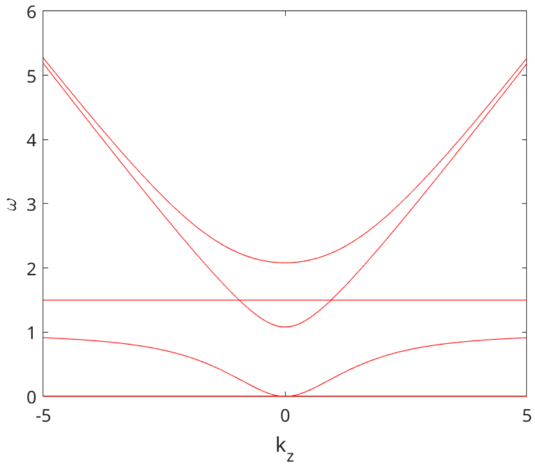
To summarize we have found the eigenvectors:

$$\Psi_p = (\hat{z}, i\hat{z}, 0)$$

$$\Psi_R = \left(-i\frac{\omega_p}{\Omega + \omega_R} e_-, e_-, i\frac{k}{\omega_R} e_-\right)$$

$$\Psi_{L^\pm} = \left(i\frac{\omega_p}{\Omega - \omega_{L^\pm}} e_+, e_+, -i\frac{k}{\omega_{L^\pm}} e_+\right)$$

corresponding to the positive eigenvalues $(\omega_p, \omega_{L_-}, \omega_R, \omega_{L^+})$. As we will see in Section 4 these eigenvectors and their respective orderings will be essential in calculating topological invariants.



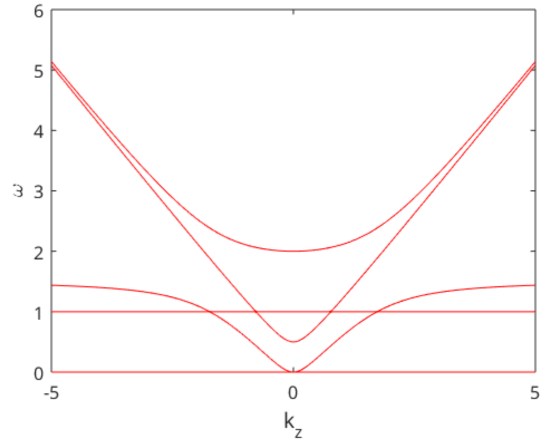




Figure 1: Eigenvalues plotted as a function of $k_z$ for $k_\perp = 0$. Left shows $(\Omega, \omega_p, k_z) = (1, 1.5, 1)$ and right shows $(\Omega, \omega_p, k_z) = (1.5, 1, 1)$. As discussed above the lower frequency L-wave starts at 0 and converges to $\Omega$ as $k_z \to \infty$. Therefore we see illustrated that there are two band crossings $(\omega_-, \omega_+)$ if $\Omega > \omega_p$ and one $(\omega_+)$ if $\Omega < \omega_p$.

### 3.2 Case: $k_\perp \to \infty$

Finding limiting eigenvectors as $k \to \infty$ will also prove to be essential for calculating topological invariants. We will analyze this situation by breaking $H$ into:

$$H = r \begin{bmatrix} -\frac{i\Omega}{r}\hat{z}\times & -\frac{i\omega_p}{r} & 0 \\ \frac{i\omega_p}{r} & 0 & -\mathbf{k}_r\times \\ 0 & \mathbf{k}_r\times & 0 \end{bmatrix} = rH_r$$

$$\mathbf{k}_r\times = \hat{\mathbf{k}}_\perp \times + \frac{1}{r}k_z\hat{z}\times$$

where $\hat{\mathbf{k}}_\perp$ is the unit vector in the $\mathbf{k}_\perp$ direction. From this decomposition we can see that $|\mathbf{k}_\perp| = r$ so as $r \to \infty$, $k_\perp \to \infty$. If we find the eigenvalue decomposition for $H_r$:

$$H_r = V_r \Lambda_r V_r^*$$

then we can see that the eigenvalue decomposition for $H$ is:

$$H = rH_r = V_r(r\Lambda_r)V_r^*$$

so $H$ and $H_r$ share eigenvectors for all $r$ and the eigenvalues of $H$ are $r$ times the eigenvalues of $H_r$. Since the eigenvectors are shared, we will focus on calculating the eigenvectors of $H_r$ as $r \to \infty$.

Denote the eigenvalues of $H$ as $\omega$ and the associated eigenvalue of $H_r$ as $\omega_r = \omega/r$. As before, we will be concerned with the positive eigenvalues and associated eigenvectors. With this in mind, we will focus on three cases for the eigenvectors of $H_r$. If $\omega$ is an eigenvector of $H$, then as $k_\perp \to \infty$ either $\omega \to \infty$, $\omega \to \bar{\omega}$ for some $0 < \bar{\omega} < \infty$, or $\omega \to 0$. If $\omega_r = \omega/r$

is the associated eigenvalue of $H_r$ then this corresponds to $\omega_r = c$ for some $0 < c < \infty$, $\omega_r = \bar{\omega}/r$, and $\omega_r = 0$ respectively.

First take the case that $\omega_r$ is just some positive constant. Then:

$$\begin{bmatrix} -\frac{i\Omega}{r}\hat{z}\times & -\frac{i\omega_p}{r}I & 0 \\ \frac{i\omega_p}{r}I & 0 & -\mathbf{k}_r\times \\ 0 & \mathbf{k}_r\times & 0 \end{bmatrix} \begin{bmatrix} v \\ E \\ B \end{bmatrix} = \omega_r \begin{bmatrix} v \\ E \\ B \end{bmatrix}$$

Taking $r \to \infty$ we can see from the first line that $v = 0$. The second and third lines then yield:

$$-\hat{\mathbf{k}}_\perp \times B = \omega_r E$$
$$\hat{\mathbf{k}}_\perp \times E = \omega_r B$$

A natural guess would be that either $E = \hat{z}$ or $B = \hat{z}$ which is perpendicular to $k_\perp$. Starting with $E = \hat{z}$ we get:

$$B = \frac{1}{\omega_r}\hat{\mathbf{k}}_\perp \times \hat{z}$$
$$-\frac{1}{\omega_r}\hat{\mathbf{k}}_\perp \times (\hat{\mathbf{k}}_\perp \times \hat{z}) = \frac{1}{\omega_r}\hat{z} = \omega_r \hat{z}$$
$$\Rightarrow \omega_r = 1$$

Similarly for $B = \hat{z}$:

$$-\hat{\mathbf{k}}_\perp \times \hat{z} = \hat{z} \times \hat{\mathbf{k}}_\perp = \omega_r E$$
$$\frac{1}{\omega_r}\hat{\mathbf{k}}_\perp \times (\hat{z} \times \hat{\mathbf{k}}_\perp) = \frac{\hat{z}}{\omega_r} = \omega_r \hat{z}$$
$$\Rightarrow \omega_r = 1$$

Therefore the two largest eigenvalues of $H$ are $\omega_{3,4} = r \cdot 1 = k_\perp \to \infty$ with eigenvectors:

$$\Psi_3 = (0, \hat{z}, \hat{\mathbf{k}}_\perp \times \hat{z})$$
$$\Psi_4 = (0, \hat{\mathbf{k}}_\perp \times \hat{z}, -\hat{z})$$

For our purposes the ordering on $\Psi_3, \Psi_4$ is immaterial as will be made apparent in Seciton 4. Next consider the case that $\omega_r = \bar{\omega}/r$ for some $0 < \bar{\omega} < \infty$. Plugging this in yields:

$$\begin{bmatrix} -\frac{i\Omega}{r}\hat{z}\times & -\frac{i\omega_p}{r}I & 0 \\ \frac{i\omega_p}{r}I & 0 & -\mathbf{k}_r\times \\ 0 & \mathbf{k}_r\times & 0 \end{bmatrix} \begin{bmatrix} v \\ E \\ B \end{bmatrix} = \frac{\bar{\omega}}{r} \begin{bmatrix} v \\ E \\ B \end{bmatrix}$$

Expanding the third row we get:

$$\hat{\mathbf{k}}_\perp \times E + \frac{k_z}{r}\hat{z} \times E = \frac{\bar{\omega}}{r}B$$

Since the $\hat{\mathbf{k}}_\perp \times E$ is the only term that does not go to zero we must have $E \to \hat{k}_\perp$ or $E \to 0$ as $r \to \infty$. Assuming for now that $E = \hat{k}_\perp$ from the second line we get:

$$i\frac{\omega_p}{r}v - \frac{k_z}{r}\hat{z} \times B - \hat{\mathbf{k}}_\perp \times B = \frac{\bar{\omega}}{r}E$$

Again as the only term that doesn't go to zero is $-\hat{\mathbf{k}}_\perp \times B$ we must have that $B = 0$ or $B = \hat{\mathbf{k}}_\perp$. Now the first line give:

$$-\frac{i\Omega}{r}\hat{z} \times v - i\frac{\omega_p}{r}E = \frac{\bar{\omega}}{r}v \Rightarrow$$

$$i\Omega\hat{z} \times v - i\omega_p E = \bar{\omega}v$$

Here we will make use of the susceptibility tensor $\chi E = v$ where $\chi$ is defined with $\omega \to \bar{\omega}$. This is possible because in this form the first row is identical to the first row of (2) with $\omega \to \bar{\omega}$. Therefore:

$$v = \chi E = \frac{\omega_p}{\Omega^2 - \bar{\omega}^2}\begin{bmatrix} i\bar{\omega} & \Omega & 0 \\ -\Omega & i\bar{\omega} & 0 \\ 0 & 0 & -i(\Omega^2 - \bar{\omega}^2)/\bar{\omega} \end{bmatrix}\begin{pmatrix} \hat{k}_x \\ \hat{k}_y \\ 0 \end{pmatrix}$$

$$= \frac{\omega_p}{\Omega^2 - \bar{\omega}^2}\begin{pmatrix} i\bar{\omega}\hat{k}_x + \Omega\hat{k}_y \\ i\bar{\omega}\hat{k}_y - \Omega\hat{k}_x \\ 0 \end{pmatrix}$$

$$= \frac{\omega_p}{\Omega^2 - \bar{\omega}^2}(i\bar{\omega}\hat{\mathbf{k}}_\perp + \Omega\hat{\mathbf{k}}_\perp \times \hat{z})$$

Therefore we have the eigenvalue $\omega_2 = r\omega_r = \bar{\omega}$ with eigenvector:

$$\Psi_2 = \left(\frac{\omega_p}{\Omega^2 - \bar{\omega}^2}(i\bar{\omega}\hat{\mathbf{k}}_\perp - \Omega\hat{\mathbf{k}}_\perp \times \hat{z}), \hat{\mathbf{k}}_\perp, 0\right)$$

Clearly we have that $\omega_2 = \bar{\omega} < \omega_3, \omega_4$ and since the only case left is $\omega_r = 0$ the remaining positive eigenvalue $\omega_1$ (whose limit is 0) obeys $\omega_1 < \omega_2$.

Finally we will find the zero eigenvectors. Substituting $\omega_r = 0$ and $r \to \infty$ we get from the last two lines:

$$\hat{\mathbf{k}}_\perp \times E = 0$$

$$-\hat{\mathbf{k}}_\perp \times B = 0$$

So again $E = 0$ or $\hat{k}_\perp$ and $B = 0$ or $\hat{k}_\perp$. From the first line we get:

$$-\Omega\hat{z} \times v = \omega_p E$$

so if $E = 0$ then $v = 0$ or $v = \hat{z}$. This recovers (for $E = 0$ and $B = \hat{\mathbf{k}}_\perp$) our universal zero eigenvector $\Psi_0 = (0, 0, \hat{\mathbf{k}}_\perp)$. If $B = 0$ and $E = \hat{k}_\perp$ then we get from the first line:

$$-\sigma\hat{z} \times v = \hat{\mathbf{k}}_\perp$$

where we have defined $\sigma = |\Omega|/\omega_p$. To ensure a basis for the whole null space we will consider that $v = a\hat{\mathbf{k}}_\perp \times \hat{z} \pm ib\hat{z}$ so that:

$$-\sigma\hat{z} \times v = -\sigma a\hat{\mathbf{k}}_\perp = \hat{\mathbf{k}}_\perp \Rightarrow a = -\frac{1}{\sigma}$$

Therefore we have found two zero eigenvectors:

$$\Psi_0^{(1)} = (\hat{\mathbf{k}}_\perp \times \hat{z} + i\sigma b\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0)$$
$$\Psi_0^{(2)} = (\hat{\mathbf{k}}_\perp \times \hat{z} - i\sigma b\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0)$$

Requiring that $\Psi_0^{(2)}$ and $\Psi_0^{(3)}$ are orthogonal we get:

$$\Psi_0^{*(2)}\Psi_0^{(3)} = 1 - \sigma^2 b^2 + \sigma^2 = 0 \Rightarrow$$

$$b = \sqrt{1 + \frac{1}{\sigma^2}} = \frac{\sqrt{1+\sigma^2}}{\sigma}$$

so we have that:

$$\Psi_0^{(1)} = (\hat{\mathbf{k}}_\perp \times \hat{z} + i\sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0)$$
$$\Psi_0^{(2)} = (\hat{\mathbf{k}}_\perp \times \hat{z} - i\sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0)$$

Here, although we have some freedom to choose the basis of the 0 eigenvectors due to three-fold degeneracy, we were careful to choose the basis most consistent with the general case. $\Psi_0$ is clearly the universal 0 eigenvector and we have chosen $\Psi_0^{(1)}$ and $\Psi_0^{(2)}$ to obey the $\pm\omega$ symmetry as described in Section 2 because they are in general limits of two non-zero eigenvectors with $\pm\omega$ symmetry. Although in principle one of these eigenvectors belongs to a positive eigenvalue and one to a negative one, we will see in the next section that these vectors both produce trivial topological invariants and so denote $\Psi_1 = \Psi_0^{(1)}$ as the eigenvector with the smallest eigenvalue (whose limit is 0 as $k_\perp \to \infty$).

It is apparent now that all eigenvector pairs are orthogonal to one another except $\Psi_2$ with $\Psi_{\pm 1}$. We also did not specify an eigenvalue for $\bar{\omega}$ however so we still have one free variable. Taking the inner product:

$$\Psi_2^*\Psi_{\pm 1} = \left(\frac{\omega_p}{\Omega^2 - \bar{\omega}^2}(-i\bar{\omega}\hat{\mathbf{k}}_\perp - \Omega\hat{\mathbf{k}}_\perp \times \hat{z}), \hat{\mathbf{k}}_\perp, 0\right) \cdot \left(\hat{\mathbf{k}}_\perp \times \hat{z} \pm i\sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0\right)$$

$$= -\frac{\Omega\omega_p}{\Omega^2 - \bar{\omega}^2} - \sigma = 0 \Rightarrow$$
$$\bar{\omega}^2 = \omega_p^2 + \Omega^2 = \omega_{uh}^2$$

Plugging this result in and summarizing, the eigenvectors and eigenvalues of $H$ as $k_\perp \to \infty$ are:

$$\omega_1 = 0, \quad \omega_2 = \omega_{uh} = \sqrt{\omega_p^2 + \Omega^2}, \quad \omega_{3,4} = k_\perp(\to \infty)$$

$$\Psi_1 = (\hat{\mathbf{k}}_\perp \times \hat{z} + i\sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0)$$

$$\Psi_2 = \left(-\sigma\hat{\mathbf{k}}_\perp \times \hat{z} - i\,\mathrm{sgn}(\omega_p)\sqrt{1+\sigma^2}\hat{\mathbf{k}}_\perp, \hat{\mathbf{k}}_\perp, 0\right)$$

$$\Psi_3 = (0, \hat{z}, \hat{\mathbf{k}}_\perp \times \hat{z}), \quad \Psi_4 = (0, \hat{\mathbf{k}}_\perp \times \hat{z}, -\hat{z})$$

### 3.3 Case: $k_z = 0$

Like $\Omega = 0$, the $k_z = 0$ case produces coincident bands, and so is also interesting to study on its own. Calculation of eigenvectors utilizes the plasma wave equation and largely mirrors the $k_\perp = 0$ case. Full calculations are shown in Appendix B, and the eigenvalues and eigenvectors are as follows:

$$\omega_1 = 0$$

$$\Psi_1^{(2)} = (k_\perp^2 \hat{z}, 0, -i\omega_p \mathbf{k}_\perp \times \hat{z}) \quad \Psi_1^{(3)} = (\mathbf{k}_\perp \times \hat{z}, -\sigma \mathbf{k}_\perp, i\omega_p \hat{z})$$

$$\omega_3 = \sqrt{k_\perp^2 + \omega_p^2}$$

$$\Psi_3 = (-i\omega_p \hat{z}, \omega_3 \hat{z}, \mathbf{k}_\perp \times \hat{z})$$

$$\omega_{2,4} = \frac{1}{2}\left((k^2 + \omega_h^2) \pm \sqrt{(k^2 + \omega_h^2)^2 - 4(\omega_p^4 + k^2 \omega_{uh}^2)}\right)$$

$$\Psi_{2,4} = \left(i\frac{(1-\alpha)}{\omega_p}\mathbf{k}_\perp \times \hat{z} + \frac{\beta}{\omega_p}\mathbf{k}_\perp, \frac{1}{\omega}(i\beta \mathbf{k}_\perp + \alpha \mathbf{k}_\perp \times \hat{z}), -\hat{z}\right)$$

$$\alpha = \frac{\omega^2}{k^2} = \frac{(\omega^2 - \omega_{uh}^2)\omega^2}{\omega^4 - \omega^2 \omega_h^2 + \omega_p^4}$$

$$\beta = \frac{\omega_p^2 \Omega \omega}{\omega^4 - \omega^2 \omega_h^2 + \omega_p^4}$$

Taking the limit as $k_\perp \to \infty$ reveals that $\omega_2 \to \omega_{uh}$ so indeed $\omega_1 < \omega_2 < \omega_3 < \omega_4$. We again also find three degenerate 0 eigenvectors, but here do not attempt to express two of them as limits of positive branches. We will see below that the eigenvectors $\Psi_1^{(1)}, \Psi_1^{(2)}$ are more natural choices in the $k_z = 0$ case.

### 3.4 Case: $\Omega = 0$

The non-negative eigenvalues and eigenvectors are shown below. Due to $\pm\omega$ symmetry as proved above all bands are symmetric around $\omega = 0$ so it is sufficient to consider only non-negative bands. This case contains multiple sets of degenerate eigenvalues so for concreteness we choose $k_y = 0$ which allows for explicit expressions which may be unitarily transformed for arbitrary $\mathbf{k}$ by $\mathbf{R}$ as detailed in the previous section. Detailed calculations are shown in Appendix A.

$$\omega_1 = 0 \quad \omega_2 = \omega_p \quad \omega_{3,4} = \sqrt{\omega_p^2 + k^2}$$

$$\Psi_1 = (-k(\hat{y} + i\hat{k} \times \hat{y}), 0, \omega_p(\hat{y} + i\hat{k} \times \hat{y}))$$

$$\Psi_2 = (k, ik, 0)$$

$$\Psi_3 = (\omega_p \hat{y}, i\omega_3 \hat{y}, ik \times \hat{y})$$

$$\Psi_4 = (\omega_p k \times \hat{y}, i\omega_4 k \times \hat{y}, -ik^2 \hat{y})$$

# 4 Topological Phases and Invariants

## 4.1 Chern Number Definitions

The Chern number of a particular level of the system is defined as the integral over the parameter space (in our case $\mathbf{k}_\perp$) of the Berry Curvature:

$$C_n = \frac{1}{2\pi} \int F_n(\mathbf{k}_\perp) \cdot d\mathbf{k}_\perp \tag{11}$$

The Berry curvature in a 2-d parameter space is defined as the curl of the Berry connection $A(\mathbf{k}_\perp)$:

$$\begin{aligned} A_n(\mathbf{k}_\perp) &= i\langle \Psi_n | \nabla_{k_\perp} \Psi_n \rangle \\ F_n(\mathbf{k}_\perp) &= \nabla_{\mathbf{k}_\perp} \times A_n(\mathbf{k}_\perp) \end{aligned} \tag{12}$$

Plugging this definition in results in the more compact form for the Berry curvature:

$$F_n(\mathbf{k}_\perp) = -2\ \mathrm{Im}\langle \partial_{k_x} \Psi_n | \partial_{k_y} \Psi_n \rangle \tag{13}$$

We can also use Stokes theorem to derive the following definition, which is especially useful in the continuum case [1]:

$$\int_S \nabla_{\mathbf{k}_\perp} \times A_n(\mathbf{k}_\perp) \cdot d\mathbf{k}_\perp = \oint_{\partial S} A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} \Rightarrow$$

$$C_n = \frac{1}{2\pi} \left[ \oint_{k\to\infty} A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} - \oint_{k\to 0} A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} \right] \tag{14}$$

Finally, assuming that our eigenvectors form an orthonormal basis we have another form of the Berry Curvature which avoids taking derivatives of the eigenvectors themselves:

$$F_n(\mathbf{k}) = -2\mathrm{Im} \left[ \sum_{m\neq n} \frac{\langle \Psi_n | \partial_{k_x} H | \Psi_m \rangle \langle \Psi_m | \partial_{k_y} H | \Psi_n \rangle}{(\omega_n - \omega_m)^2} \right] \tag{15}$$

For full derivation of the above quantities and deeper analysis and discussion see [12]. When the dependence of the eigenvectors on $\mathbf{k}_\perp$ is straightforward ($\Omega = 0$) using (11) and (13) is is sufficient. For more complex dependence of $\Psi_n$ on $\mathbf{k}_\perp$ (all other cases) (14) will be more useful, and for numerical calculations (15) is the most useful (having transferred the derivatives to $H$ allows numerical calculation of $\Psi_n$'s), though we focus on analytical calculations here. Note that since our domain of integration is $\mathbb{R}^2$ our Chern numbers as defined are not required to be integers, as we discuss further in upcoming sections.

Finally, the quantity we are most interested in is the gap Chern number. Given a transition between two topological phases labeled $\pm$, we define the gap Chern number as:

$$\Delta C_n^{\pm} = \sum_{j\leq n} C_n^+ - \sum_{j\leq n} C_n^-$$

The bulk-edge correspondence (BEC) is a general principle which states that the number of topologically protected edge states in a global band gap between bands $n$ and $n+1$

at an interface between two insulating bulk phases should be equal to $\Delta C_n^{\pm}$. Clearly for this conjecture to be true, we must have that $\Delta C_n^{\pm} \in \mathbb{Z}$. Although in settings where the Brillouin zone is compact it is guaranteed that all Chern numbers $C_n$ are integers, but in our continuous setting we will see that some modification is needed to ensure gap Chern numbers are integers.

## 4.2 Rotational Invariance

Next, by leveraging the $k_x/k_y$ plane symmetry, we can find a more useful form of (14). From Section 3.4 and (15) we can clearly see (since $R^T R = I$) that the Berry Curvature in our case is rotationally symmetric in the $x/y$ plane. We can also use the rotational symmetry of eigenvectors in the $k_x/k_y$ plane to derive a more direct equation for $C_n$ using (14). First, parametrize the $\mathbf{k}_\perp$ plane in terms of $(k, \theta)$, radial and polar coordinates in the $\mathbf{k}_\perp$ plane:

$$k_x = k\cos\theta$$

$$k_y = k\sin\theta$$

Then we can write the contour integral in (14) as:

$$\oint A_n(\mathbf{k}_\perp)\cdot \mathbf{dl} = i\oint \Psi_n^* D_{\mathbf{k}_\perp}\Psi_n \cdot \mathbf{dl} = ik\int_0^{2\pi} (\Psi_n^* D_{\mathbf{k}_\perp}\Psi_n)\cdot\hat{\theta}\, d\theta$$

$$= ik\int_0^{2\pi} \begin{pmatrix}\Psi_n^*\partial_{k_x}\Psi_n, & \Psi_n^*\partial_{k_y}\Psi_n\end{pmatrix}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} d\theta$$

$$= ik\int_0^{2\pi} \cos\theta(\Psi_n^*\partial_{k_y}\Psi_n) - \sin\theta(\Psi_n^*\partial_{k_x}\Psi_n) d\theta$$

We can calculate:

$$\partial_{k_x}\Psi_n = \frac{\partial k}{\partial k_x}\partial_k\Psi_n + \frac{\partial\theta}{\partial k_x}\partial_\theta\Psi_n = \frac{k\cos\theta\partial_k\Psi_n - \sin\theta\partial_\theta\Psi_n}{k}$$

$$\partial_{k_y}\Psi_n = \frac{\partial k}{\partial k_y}\partial_k\Psi_n + \frac{\partial\theta}{\partial k_y}\partial_\theta\Psi_n = \frac{k\sin\theta\partial_k\Psi_n + \cos\theta\partial_\theta\Psi_n}{k}$$

$$\Rightarrow k\left(\cos\theta(\Psi_n^*\partial_{k_y}\Psi_n) - \sin\theta(\Psi_n^*\partial_{k_x}\Psi_n)\right)$$

$$= -k\cos\theta\sin\theta\Psi_n^*\partial_k\Psi_n + \sin^2\theta\Psi_n^*\partial_\theta\Psi_n + k\cos\theta\sin\theta\Psi_n^*\partial_k\Psi_n + \cos^2\theta\Psi_n^*\partial_\theta\Psi_n$$

$$= \Psi_n^*\partial_\theta\Psi_n$$

Therefore we get:

$$\oint A_n(\mathbf{k}_\perp)\cdot\mathbf{dl} = ik\int_0^{2\pi}\cos\theta(\Psi_n^*\partial_{k_y}\Psi_n) - \sin\theta(\Psi_n^*\partial_{k_x}\Psi_n)d\theta = i\int_0^{2\pi}\Psi_n^*\partial_\theta\Psi_n d\theta$$

Now due to the rotational symmetry of $\Psi_n$ we can parametrize $\Psi_n$ as $\Psi_n = \mathbf{R}\Psi_{n0}$ where $\Psi_{n0} = \Psi_n(k, \theta = 0)$ and $\mathbf{R}$ defined as in Section 3.4. Then since $\Psi_{n0}$ is fixed we get:

$$\Psi_n^*\partial_\theta\Psi_n = \Psi_{n0}^*\mathbf{R}^T\partial_\theta(\mathbf{R}\Psi_{n0}) = \Psi_{n0}^*\mathbf{R}^T\partial_\theta(\mathbf{R})\Psi_{n0}$$

Remembering from Section 3.4 that:

$$\mathbf{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we get that

$$\partial_\theta \mathbf{R} = \begin{bmatrix} \partial_\theta R & 0 & 0 \\ 0 & \partial_\theta R & 0 \\ 0 & 0 & \partial_\theta R \end{bmatrix}$$

with

$$\partial_\theta R = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R = [\hat{z}\times]R = R\hat{z}\times$$

where in the last equality we used the fact from 3.4 that $R$ and $\hat{z}\times$ commute. Denoting the matrix:

$$\mathbf{Z}\times = \begin{bmatrix} \hat{z}\times & 0 & 0 \\ 0 & \hat{z}\times & 0 \\ 0 & 0 & \hat{z}\times \end{bmatrix}$$

we get:

$$\Psi_n^* \partial_\theta \Psi_n = \Psi_{n0}^* \mathbf{R}^T \mathbf{R} \mathbf{Z} \times \Psi_{n0} = \Psi_{n0}^* \mathbf{Z} \times \Psi_{n0}$$

Therefore we get:

$$\oint A_n(\mathbf{k}_\perp) \cdot \mathbf{dl} = i \int_0^{2\pi} \Psi_n^* \partial_\theta \Psi_n d\theta = i \int_0^{2\pi} \Psi_{n0}^* \mathbf{Z} \times \Psi_{n0} d\theta$$

Since $\Psi_{n0}$ is fixed for a given $k$ we can plug this into (14) to get:

$$C_n = \lim_{k\to\infty} i\Psi_{n0}^*(\mathbf{Z} \times \Psi_{n0}) - \lim_{k\to 0} i\Psi_{n0}^*(\mathbf{Z} \times \Psi_{n0}) \tag{16}$$

Note that this result holds for any 2-dimensional parameter space with rotational symmetry (e.g. any system where the final result of 3.4 holds), see also [1].

## 4.3 Chern Number Symmetry

Suppose that we have calculated a Chern number $C_n$ for a positive band with eigenvalue $\omega_n$ with positive $\Omega$. Supposing that the eigenvector for this band is simply $(v, E, B)^T$, we know that the Chern number is given by (14) with:

$$\Psi_{n0}^* \mathbf{Z} \times \Psi_{n0} = v^*(\hat{z} \times v) + E^*(\hat{z} \times E) + B^*(\hat{z} \times B)$$

Now suppose that we want to find the Chern number $C_{-n}$ for the associated negative band $-\omega_n$. From Section 4.1 we know the associated eigenvector is $(v^*, E^*, -B^*)$ so we get:

$$\Psi^*_{-n0}\mathbf{Z}\times\Psi_{-n0} = v^T(\hat{z}\times v^*)+E^T(\hat{z}\times E^*)+B^T(\hat{z}\times B^*) = (v^*(\hat{z}\times v))^*+(E^*(\hat{z}\times E))^*+(B^*(\hat{z}\times B))^*$$

$$= (\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0})^*$$

Assuming that $C_n$ is a real number, we know that $\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0}$ must be purely imaginary, so we get:

$$\Psi^*_{-n0}\mathbf{Z}\times\Psi_{-n0} = (\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0})^* = -\Psi^*_{n0}\mathbf{Z}\times\Psi_{n0}$$

$$\Rightarrow C_{-n} = -C_n$$

from (14). If we repeat this analysis instead for $C_{-\Omega n}$, the Chern number the $\omega_n$ band reflected across $\Omega = 0$ we get an identical result by using the results of Section 4.2, so $C_n(-\Omega) = -C_n$. Applying the results of Section 4.3 and 4.4 it's clear that reflecting $k_z \to -k_z$ gives an identical Chern number, $C_{-k_z n} = C_n$. Therefore, it's sufficient to calculate Chern numbers for bands that have $\omega, \Omega, k_z > 0$.

## 4.4 Chern Number Calculations

We now have all the tools necessary to calculate Chern numbers for all bands. First the general case is presented, but the $B = 0$ and $k_z = 0$ are also interesting to consider by themselves and are shown as well.

### 4.4.1 General Chern Numbers

Here we calculate Chern numbers for non-negative bands assuming that $\Omega, k_z > 0$. The symmetry derived in Section 6.1 generalizes these Chern numbers to any $\Omega, k_z \neq 0$. To use (16) we need only calculate 8 values, two for each positive eigenvalue $\omega_n$, to calculate $C_n$:

$$\lim_{k_\perp\to 0}\frac{i\Psi^*_n(\mathbf{Z}\times\Psi_n)}{|\Psi_n|^2}$$

and

$$\lim_{k_\perp\to \infty}\frac{i\Psi^*_n(\mathbf{Z}\times\Psi_n)}{|\Psi_n|^2}$$

for $n \in \{1, 2, 3, 4\}$. It's easy to see that:

$$i\Psi^*_0(\mathbf{Z}\times\Psi_0) = i(0,0,\mathbf{k})\mathbf{Z}\times\begin{pmatrix}0\\0\\\mathbf{k}\end{pmatrix} = i\mathbf{k}\cdot(\hat{z}\times\mathbf{k}) = 0$$

both as $k_\perp \to 0$ and $k_\perp \to \infty$ so $C_0 = 0$. We have already computed the eigenvectors for $k_\perp = 0$ and as $k_\perp \to \infty$ so for the positive bands we can simply substitute our results from Sections 3.1 and 3.2 respectively. First for $k_\perp = 0$ we get:

$$i\Psi^*_p(\mathbf{Z}\times\Psi_p) = i(\hat{z}, -i\hat{z}, 0)\begin{pmatrix}\hat{z}\times\hat{z}\\ i\hat{z}\times\hat{z}\\ 0\end{pmatrix} = 0$$

$$i\Psi_R^*(\mathbf{Z}\times\Psi_R) = i\left(i\frac{\omega_p}{\Omega+\omega_R}e_-^*, e_-^*, -i\frac{k}{\omega_R}e_-^*\right)\begin{pmatrix}-i\frac{\omega_p}{\Omega+\omega_R}\hat{z}\times e_- \\ \hat{z}\times e_- \\ i\frac{k}{\omega_R}\hat{z}\times e_-\end{pmatrix}$$

$$= i\left(1+\frac{\omega_p^2}{(\Omega+\omega_R)^2}+\frac{k^2}{\omega_R^2}\right)e_-^*(\hat{z}\times e_-)$$

We can also evaluate:

$$e_-^*(\hat{z}\times e_-) = (1, i, 0)\begin{pmatrix}i\\1\\0\end{pmatrix} = 2i$$

Therefore:

$$i\Psi_R^*(\mathbf{Z}\times\Psi_R) = 2\left(\left(1+\frac{\omega_p^2}{(\Omega+\omega_R)^2}\right)+\frac{k^2}{\omega_R^2}\right)$$

Also notice that

$$e_-^*e_- = 2$$

so we have:

$$|\Psi_R|^2 = 2\left(\left(1+\frac{\omega_p^2}{(\Omega-\omega_R)^2}\right)+\frac{k^2}{\omega_R^2}\right)$$

Therefore:

$$\frac{i\Psi_R^*(\mathbf{Z}\times\Psi_R)}{|\Psi_R|^2} = -1$$

Finally for $\omega_L$ we get:

$$i\Psi_L^*\mathbf{Z}\times\Psi_L = i\left(-i\frac{\omega_p}{\Omega-\omega_L}e_+^*, e_+^*, i\frac{k}{\omega_L}(\hat{z}\times e_+)^*\right)\begin{pmatrix}i\frac{\omega_p}{\Omega-\omega_L}\hat{z}\times e_+ \\ \hat{z}\times e_+ \\ -i\frac{k}{\omega_L}e_+\end{pmatrix}$$

$$= i\left(\left(1+\frac{\omega_p^2}{(\Omega-\omega_L)^2}+\frac{k^2}{\omega_L^2}\right)e_+^*(\hat{z}\times e_+)\right)$$

Once again calculate:

$$e_+^*(\hat{z}\times e_+) = (1, -i, 0)\begin{pmatrix}-i\\1\\0\end{pmatrix} = -2i$$

Therefore:

$$i\Psi_L^*\mathbf{Z}\times\Psi_L = 2\left(\left(1+\frac{\omega_p^2}{(\Omega+\omega_L)^2}\right)+\frac{k^2}{\omega_L^2}\right)$$

We can also see that $e_-^*e_- = 2$ so that:

$$\frac{i\Psi_L^*\mathbf{Z}\times\Psi_L}{|\Psi_L|^2} = 1$$

Now for the $k_\perp \to \infty$ case. We can see that $\Psi_3$, $\Psi_4$ are real-valued. It's straightforward to verify that if $\Psi \in \mathbb{R}^9$:

$$\Psi^*(\mathbf{Z}\times\Psi) = \Psi^T(\mathbf{Z}\times\Psi) = 0$$

For $\Psi_1$ we get:

$$\Psi_1^*(\mathbf{Z}\times\Psi_1) = (\hat{\mathbf{k}}_\perp \times \hat{z} + i\sqrt{1+\sigma^2}\hat{z}, -\sigma\hat{\mathbf{k}}_\perp, 0)\cdot(\hat{\mathbf{k}}_\perp, -\sigma\hat{z}\times\hat{\mathbf{k}}_\perp, 0) = 0$$

Since $\hat{z}\times\hat{z} = 0$ we see here that our choice of sign in the $i\sqrt{1+\sigma^2}\hat{z}$ term was in fact immaterial for calcuation of Chern numbers. Now consider $\Psi_2$:

$$i\Psi_2^*(\mathbf{Z}\times\Psi_2) = i\left(-\sigma\hat{\mathbf{k}}_\perp\times\hat{z} + i\sqrt{1+\sigma^2}\hat{\mathbf{k}}_\perp, \hat{\mathbf{k}}_\perp, 0\right)\begin{pmatrix} -\sigma\hat{\mathbf{k}}_\perp + i\sqrt{1+\sigma^2}\ \hat{\mathbf{k}}_\perp\times\hat{z} \\ \hat{\mathbf{k}}_\perp\times\hat{z} \\ 0 \end{pmatrix}$$

$$= 2\sigma\sqrt{1+\sigma^2}$$

We can also calculate:

$$|\Psi_2|^2 = 1+\sigma^2+\sigma^2+1 = 2(\sigma^2+1)$$

Therefore:

$$i\frac{\Psi_2^*\mathbf{Z}\times\Psi_2}{|\Psi_2|^2} = \frac{2\sigma\sqrt{1+\sigma^2}}{2(1+\sigma^2)} = \frac{\sigma}{\sqrt{1+\sigma^2}}$$

Now we have all 8 quantities needed to calculate all Chern numbers, and need only match eigenvectors at $k_\perp\to\infty$ to eigenvectors at $k_\perp = 0$ to obtain Chern numbers from (16):

$$C_n = \lim_{k_\perp\to\infty} i\frac{\Psi_n^*\mathbf{Z}\times\Psi_n}{|\Psi_n|^2} - \lim_{k_\perp\to 0} i\frac{\Psi_n^*\mathbf{Z}\times\Psi_n}{|\Psi_n|^2}$$

As noted in Section 3.2 we have $\omega_{L^-} < \omega_R < \omega_{L^+}$ for all values of $k_z$ when $k_\perp = 0$, and rely on extensive numerical simulations to verify that band crossings happen only when $k_\perp = 0, k_z = 0, \Omega = 0$. We also calculated that the ordering of the eigenvalue $\omega_p$ with respect to $\omega_{L^-}, \omega_R, \omega_{L^+}$ is determined by the critical values $\omega_- = \omega_p = \omega_{L_-}$ and $\omega_+ = \omega_p = \omega_R$. This allows us to define three distinct topological phases for $k_z, \Omega, \omega_p > 0$:

| Phase | Eigenvalue Ordering $(\omega_1, \omega_2, \omega_3, \omega_4)$ |
|---|---|
| I | $(\omega_p, \omega_{L^-}, \omega_R, \omega_{L^+})$ |
| II | $(\omega_{L^-}, \omega_p, \omega_R, \omega_{L^+})$ |
| III | $(\omega_{L^-}, \omega_R, \omega_p, \omega_{L^+})$ |

Since we assume that bands do not cross if $k_\perp \neq 0$ then this allow us to pair the eigenvectors $\Psi_p, \Psi_{L^-}, \Psi_R, \Psi_{L^+}$ at $k_\perp = 0$ with the eigenvectors $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ at $k_\perp\to\infty$ defining Chern numbers as follows:

| Phase | Eigenvector Pairing | Chern Numbers $(C_1, C_2, C_3, C_4)$ |
|---|---|---|
| I | $\Psi_p\to\Psi_1,\ \Psi_{L^-}\to\Psi_2,\ \Psi_R\to\Psi_3,\ \Psi_{L^+}\to\Psi_4$ | $\left(0, \frac{\sigma}{\sqrt{\sigma^2+1}} - 1, 1, -1\right)$ |
| II | $\Psi_{L^-}\to\Psi_1,\ \Psi_p\to\Psi_2,\ \Psi_R\to\Psi_3,\ \Psi_{L^+}\to\Psi_4$ | $\left(-1, \frac{\sigma}{\sqrt{\sigma^2+1}}, 1, -1\right)$ |
| III | $\Psi_{L^-}\to\Psi_1,\ \Psi_R\to\Psi_2,\ \Psi_p\to\Psi_3,\ \Psi_{L^+}\to\Psi_4$ | $\left(-1, 1+\frac{\sigma}{\sqrt{\sigma^2+1}}, 0, -1\right)$ |

We have assumed that $\omega_p, \Omega, k_z > 0$ in the above derivation but by the symmetries outlined in the previous section we know that refleciton of $k_z \to -k_z$ leaves $C_n$ invariant and for $\Omega \to -\Omega$ $C_n \to -C_n$. As $n_e$ is fundamentally positive we consider only positive values of $\omega_p$. Therefore with the exception of the (possibly) singular cases $k_z = 0, \Omega = 0$ described below, the following are Chern numbers for all parameter values:

| Phase | $(C_1, C_2, C_3, C_4)$ | Phase | $(C_1, C_2, C_3, C_4)$ |
|---|---|---|---|
| I | $\left(0, \frac{\sigma}{\sqrt{\sigma^2+1}} - 1, 1, -1\right)$ | I⁻ | $\left(0, -\frac{\sigma}{\sqrt{\sigma^2+1}} + 1, -1, 1\right)$ |
| II | $\left(-1, \frac{\sigma}{\sqrt{\sigma^2+1}}, 1, -1\right)$ | II⁻ | $\left(1, -\frac{\sigma}{\sqrt{\sigma^2+1}}, -1, 1\right)$ |
| III | $\left(-1, 1 + \frac{\sigma}{\sqrt{\sigma^2+1}}, 0, -1\right)$ | III⁻ | $\left(1, -1 - \frac{\sigma}{\sqrt{\sigma^2+1}}, 0, 1\right)$ |

where the $-$ superscript denotes the transformation $\Omega \to -\Omega$.

It is a well-studied fact of topological invariants in continuous media that unless $\lim_{k_\perp \to \infty} \Psi_n(k) = \Psi_{n\infty}$ independently of $\hat{k}_\perp$, the Chern theorem does not apply and thus Chern numbers are not guaranteed to be integers. In our case note that in fact none of the eigenvectors are independent of $\hat{k}_\perp$ as $k_\perp \to \infty$, however, for all except $\Psi_1$ $\lim_{k_\perp \to \infty} A_n(k) = 0$ for our chosen gauge, so $A_n(k)$ is in fact smooth at $k_\perp = \infty$ and $C_n$ is guaranteed to be an integer for all except band 2.

For band 2 unfortunately, $A_2(k)$ depends on $\sigma$, which is smoothly varying within each topological phase. Therefore, not only is $C_2$ not an integer, but in general it is not a topological invariant at all. One way to restore invariance is to assume that $\sigma$ is constant, or that $(\omega_p, \Omega) \to (\mu\omega_p, \mu\Omega)$ for $\mu \in \mathbb{R}$. This allows transitions between topological phases with constant $\sigma$ and therefore guarantees integer differences in Chern numbers. However, this is not stable with respect to perturbations in $\omega_p$ or $\Omega$ and therefore its physical significance may be limited. A more robust way to restore invariance in Chern numbers is through regularization as discussed below.

### 4.4.2 Regularized Chern Numbers

As we saw in the last section, the non-invariance of Chern numbers comes from the behavior of the eigenvectors at $k_\perp \to \infty$. One popular approach to address this issue is to regularize the behavior of $H$ as $k_\perp \to \infty$. The dominant approach to regularization of cold plasma, introduced in [1], is to assume that $\omega_p \to \omega_p(k)$ and as $k \to \infty$, $\omega_p \to 0$. This approach is justified by reasoning that as $k \to \infty$ the wavelength becomes much smaller than the average particle spacing in the classical sense, so that the spectral response should in fact converge to the free space Hamiltonian for electromagnetic waves. If we assume that:

$$\omega_p(k) = \frac{\bar{\omega}_p}{1 + \left(\frac{k}{\alpha}\right)^2}$$

where $\alpha \in \mathbb{R}$ is some large constant representing the cutoff wave number, the repeating the analysis in Section 3.4 we get eigenvalue/eigenvector pairs:

$$\omega_1 = 0$$

$$\Psi_1 = (i\hat{z}, -\hat{\mathbf{k}}_\perp, 0)$$
$$\omega_2 = \Omega$$
$$\Psi_2 = (\hat{\mathbf{k}}_\perp + i\hat{\mathbf{k}}_\perp \times \hat{z}, 0, 0)$$
$$\omega_{3,4} = k_\perp \ (\to \infty)$$
$$\Psi_3 = (0, \hat{z}, \hat{\mathbf{k}}_\perp \times \hat{z}) \ \ \Psi_4 = (0, \hat{\mathbf{k}}_\perp \times \hat{z}, -\hat{z})$$

Subsequently repeating the calculations in the previous section with these regularized eigenvectors gives Chern numbers:

| Region | $(C_1, C_2, C_3, C_4)$ |
|---|---|
| I | $(0, 0, 1, -1)$ |
| II | $(-1, 1, 1, -1)$ |
| III | $(-1, 2, 0, -1)$ |

We can see that indeed integer Chern numbers are restored. Further, we will see numerical evidence below that at the interface between phases I and II, which is defined by the band crossing between $\omega_1$ and $\omega_2$ at $\omega_p = \omega_-$, the number of edge states matches the difference in Chern number between phases I and II for bands I and II.

However, with some further analysis we can show that this regularization is not unique considering positive values of $\Omega$ and $\omega_p$. Consider instead the regularization such that $\omega_p(k) \to \bar{\omega}_p$ and $\Omega(k) \to \bar{\Omega}$ as $k \to \infty$ so that:

$$\bar{\sigma} = \frac{\bar{\Omega}}{\bar{\omega}_p}$$

Plugging in this regularization we can simply substitute $\sigma \to \bar{\sigma}$ and the resulting Chern numbers are:

| Region | $(C_1, C_2, C_3, C_4)$ |
|---|---|
| I | $\left(0, \frac{\bar{\sigma}}{\sqrt{\bar{\sigma}^2+1}} - 1, 1, -1\right)$ |
| II | $\left(-1, \frac{\bar{\sigma}}{\sqrt{\bar{\sigma}^2+1}}, 1, -1\right)$ |
| III | $\left(-1, 1 + \frac{\bar{\sigma}}{\sqrt{\bar{\sigma}^2+1}}, 0, -1\right)$ |

Note here that $\bar{\sigma}$ is constant across all topological phases, so all gap Chern numbers are integers. Although this regularization lacks the intuitive physical interpretation of the previous one, it shows that the original regularization is not unique and for $\omega_p, \Omega > 0$ any regularization of $\omega_p$ and $\Omega$ resulting in a constant $\sigma$ is guarantees integer gap Chern numbers and in fact is consistent with the number of edge states observed as we will show below.

However, if we consider varying $\Omega$ to be negative, arbitrary regularizations are no longer valid. Consider now the interface between phase II where $\Omega = B > 0$ and phase II$^-$ where $\Omega = -B$ with $k_z$ and $\omega_p$ positive constants, and specifically the transition of band 2. Using the arbitrary regularization as above the gap Chern number for band 2 across this interface is:

$$\Delta C_2 = \frac{2\bar{\sigma}}{\sqrt{\bar{\sigma}^2 + 1}}$$

Since in general this is not an integer, we can see that arbitrary regularizations are not valid. Using the standard regularization we first considered restores $\Delta C_2 = 2$. However, further analysis shows us that even this regularization cannot be consistent with the number of edge states across this interface. Assuming that band crossings only happen at $\omega_p = \omega_\pm$, $k_z = 0$ and $\Omega = 0$ the only place where a band crossing can occur in this case is at the interface where $\Omega = 0$. However, from Section 3.4 we know that the eigenvalues of bands 1, 2, and 3 respectively are $(0, \omega_p, \sqrt{\omega_p^2 + k^2})$. Since we have assumed that $k_z > 0$ this shows that band 2 does not cross any other band at the interface between II and II$^-$ as we have defined them, hence no edge states can exist. This would contradict the edge states predicted by the gap Chern number, which is 2.

In order to restore results which may be consistent with predictions of edge states, consider instead the regularization:

$$\Omega(k) = \frac{\Omega}{1 + \left(\frac{k}{\alpha}\right)^2}$$

so that $\Omega \to 0$ as $k \to \infty$. Repeating the analysis of Section 3.4 again we get eigenvalue/eigenvector pairs:

$$\omega_1 = 0$$

$$\Psi_1 = (\hat{\mathbf{k}}_\perp \times \hat{z} + i\hat{z}, 0, 0)$$

$$\omega_2 = \omega_p$$

$$\Psi_2 = (-i\hat{\mathbf{k}}_\perp, \hat{\mathbf{k}}_\perp, 0)$$

$$\omega_{3,4} = k_\perp \ (\to \infty)$$

$$\Psi_3 = (0, \hat{\mathbf{k}}_\perp \times \hat{z}, -\hat{z}) \ \ \Psi_4 = (0, \hat{z}, \hat{\mathbf{k}}_\perp \times \hat{z})$$

These eigenvectors produce Chern numbers:

| Region | $(C_1, C_2, C_3, C_4)$ |
|---|---|
| I | $(0, -1, 1, -1)$ |
| II | $(-1, 0, 1, -1)$ |
| III | $(-1, 1, 0, -1)$ |

These results are now consistent with the phase II to II$^-$ interface described above. We will also show numerically below that these results predict the correct number of edge states for *all* topological boundaries which can be numerically analyzed.

### 4.4.3 $B_0 = 0$

Due to the straightforward expressions for eigenvectors in the $\Omega = 0$ case we can apply (12) directly to calculate Chern numbers in this case. Detailed calculations are shown in Appendix A and result in all trivial Chern numbers in this case $C_n = 0$. Therefore the $\Omega = 0$ case is not topologically relevant by itself but may serve as a non-trivial topological transition as we shall see below.

### 4.4.4 $k_z = 0$

We find that $k_z = 0$, although nominally part of region III calculated above, is topologically distinct from the general case. Again detailed calculations are left for Appendix B. For an alternate approach of $k_z = 0$ Chern number calculations see [16], where these quantities were first calculated. To summarize the results of Appendix B we have the following Chern numbers for $k_z = 0$:

$$(C_1, C_2, C_3, C_4) = \left(0, \frac{\sigma}{\sqrt{\sigma^2+1}} + 1, 0, -1\right)$$

Taking the regularization $\Omega(k) \to 0$ as $k \to \infty$ we get:

$$(C_1, C_2, C_3, C_4) = (0, 1, 0, -1)$$

Here we can see that although $k_z = 0$ in nominally part of phase III, $C_1 = -1$ in phase III whereas $C_1 = 0$ when $k_z = 0$, making this a singular case. Due to the fact that $\omega_1 = 0$ when $k_z = 0$ this singularity is not topologically relevant, however the particularity of the $k_z = 0$ case goes beyond this singularity. Writing out the full Hamiltonian we get:

$$H = \begin{bmatrix} 0 & i\Omega & 0 & -i\omega_p & 0 & 0 & 0 & 0 & 0 \\ -i\Omega & 0 & 0 & 0 & -i\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\omega_p & 0 & 0 & 0 \\ i\omega_p & 0 & 0 & 0 & 0 & 0 & 0 & k_z & -k_y \\ 0 & i\omega_p & 0 & 0 & 0 & 0 & -k_z & 0 & k_x \\ 0 & 0 & i\omega_p & 0 & 0 & 0 & k_y & -k_z & 0 \\ 0 & 0 & 0 & 0 & -k_z & k_y & 0 & 0 & 0 \\ 0 & 0 & 0 & k_z & 0 & -k_x & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_y & k_x & 0 & 0 & 0 & 0 \end{bmatrix}$$

Permuting the basis $(v_x, v_y, v_z, E_x, E_y, E_z, B_x, B_y, B_z) \to (v_x, v_y, E_x, E_y, B_z, v_z, E_z, B_x, B_y)$ the Hamiltonian becomes:

$$H = \begin{bmatrix} 0 & i\Omega & -i\omega_p & 0 & 0 & 0 & 0 & 0 & 0 \\ -i\Omega & 0 & 0 & -i\omega_p & 0 & 0 & 0 & 0 & 0 \\ i\omega_p & 0 & 0 & 0 & -k_y & 0 & 0 & 0 & k_z \\ 0 & i\omega_p & 0 & 0 & k_x & 0 & 0 & -k_z & 0 \\ 0 & 0 & -k_y & k_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\omega_p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\omega_p & 0 & k_y & -k_x \\ 0 & 0 & 0 & -k_z & 0 & 0 & k_y & 0 & 0 \\ 0 & 0 & k_z & 0 & 0 & 0 & -k_x & 0 & 0 \end{bmatrix}$$

Now if $k_z = 0$ this system becomes de-coupled into the $(v_x, v_y, E_x, E_y, B_z)$ system:

$$H_{TM} = \begin{bmatrix} 0 & i\Omega & -i\omega_p & 0 & 0 \\ -i\Omega & 0 & 0 & -i\omega_p & 0 \\ i\omega_p & 0 & 0 & 0 & -k_y \\ 0 & i\omega_p & 0 & 0 & k_x \\ 0 & 0 & -k_y & k_x & 0 \end{bmatrix}$$

which produces what are termed the Transverse Magnetic (TM) modes, and the $(v_z, E_z, B_x, B_y)$ system

$$H_{TE} = \begin{bmatrix} 0 & -i\omega_p & 0 & 0 \\ i\omega_p & 0 & k_y & -k_x \\ 0 & k_y & 0 & 0 \\ 0 & -k_x & 0 & 0 \end{bmatrix}$$

which produces what are termed Transverse Electric (TE) modes. Comparing with the eigenvectors we calculated in Appendix B we can see that the TM eigenvectors are $\Psi_{-4}$, $\Psi_{-3}$, $\Psi_{\pm 1}$, $\Psi_2$, and $\Psi_4$ with respective Chern numbers (1, -1, 0, 1, -1) and the TE eigenvectors are $\Psi_{-3}$, $\Psi_0$, $\Psi_{\mp 1}$ and $\Psi_3$ with Chern numbers all 0. We also find a global band gap between $\omega_2$ and $\omega_4$, so the de-coupling of TM and TE modes allows for the prediction of two topologically protected edge states at the interface from $\Omega = +B$ to $\Omega = -B$ when $k_z = 0$. This interface has been studied extensively in the literature [1][5][17][16]. Due to the fact that the $k_z = 0$ case is distinct from phase III, to which it nominally belongs, and topologically non-trivial, we denote new phases IV$^{\pm}$ for $\Omega > 0$ and $\Omega < 0$ respectively.

However, it is important to note that it is only this singular case $k_z = 0$ which allows us to de-couple TE and TM modes in this way. As soon as $k_z$ is perturbed (even infinitesimally) away from 0, TE modes and TM become coupled, and band 1 (previously a trivial TE mode) becomes topologically significant. Crucially, if $\Psi_3$ fills the gap between $\omega_2$ and $\omega_4$ in all cases, so as noted in [15] as soon as $\omega_3$ must be considered a part of the system these analyses fail. Therefore, the stability of the $k_z = 0$ condition is crucial when considering whether this configuration is physically realizable.

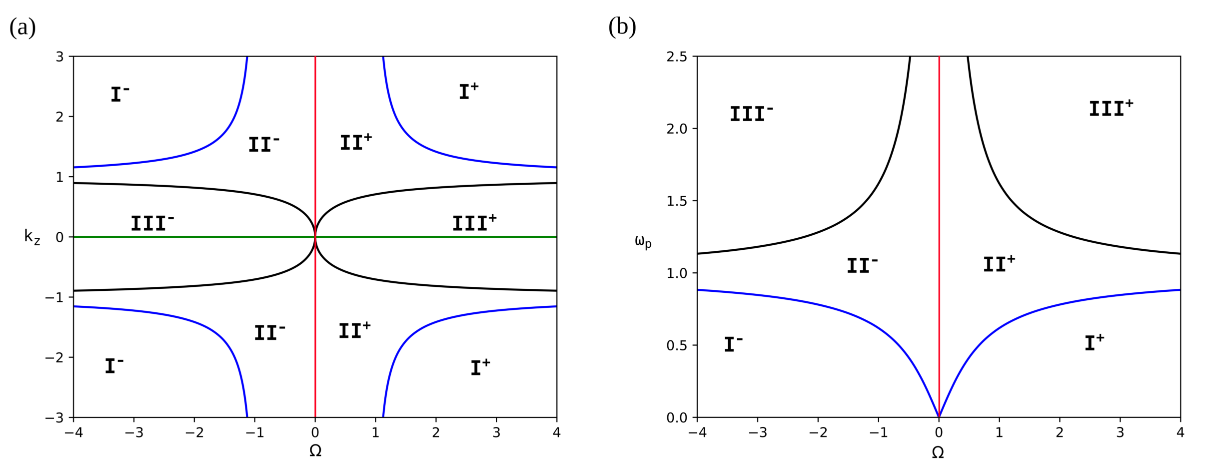



Figure 2: Topological phases plotted in the (a) $\Omega, k_z$ plane for $\omega_p = 1$ and (b) $\Omega, \omega_p$ plane for $k_z = 1$. Black lines indicate the $k_z = k_+$ and $\omega_p = \omega_+$ boundaries, blue the $k_z = k_-$ and $\omega_p = \omega_-$ boundaries, and red $\Omega = 0$ boundary. The green line on the left represents phase IV$^{\pm}$ ($k_z = 0$).

# 5 Prediction of Topological Edge States

The topological phases as described in the previous section are visualized in Figure 2. Although we treat $k_z$ as a parameter, it is still a parameter of the excitation and not the underlying plasma or material model. Therefore we consider only phase transitions for constant $k_z$. In addition, we consider only continuous transitions between adjacent phases. Finally, from Section 3.2 we have that $\omega_1 \to 0$ and $\omega_{3,4} \to \infty$ so there are only two possible band gaps between bands 1 and 2 and between bands 2 and 3. We label the associated gap Chern numbers $c_1 = \Delta C_1^{\pm}$ and $c_2 = \Delta C_2^{\pm}$, where $\pm$ denotes a continuous transition between two of the adjacent topological phases described above and shown in Figure 2. Enumerating all the possible transitions we get:

| Phase Transition | $(c_1, c_2)$ |
|---|---|
| $\mathrm{I}^{\pm} \to \mathrm{II}^{\pm}$ | $(\pm 1, 0)$ |
| $\mathrm{II}^{\pm} \to \mathrm{III}^{\pm}$ | $(0, \mp 1)$ |
| $\mathrm{II}^{-} \to \mathrm{II}^{+}$ | $(0, 0)$ |
| $\mathrm{IV}^{-} \to \mathrm{IV}^{+}$ | $(0, 2)$ |

The BEC states that these gap Chern numbers should correspond to the number of topologically protected edge states concentrated around the interface between respective bulk phases. We compare these predictions with numerical simulations of the spetra of edge Hamiltonians below.

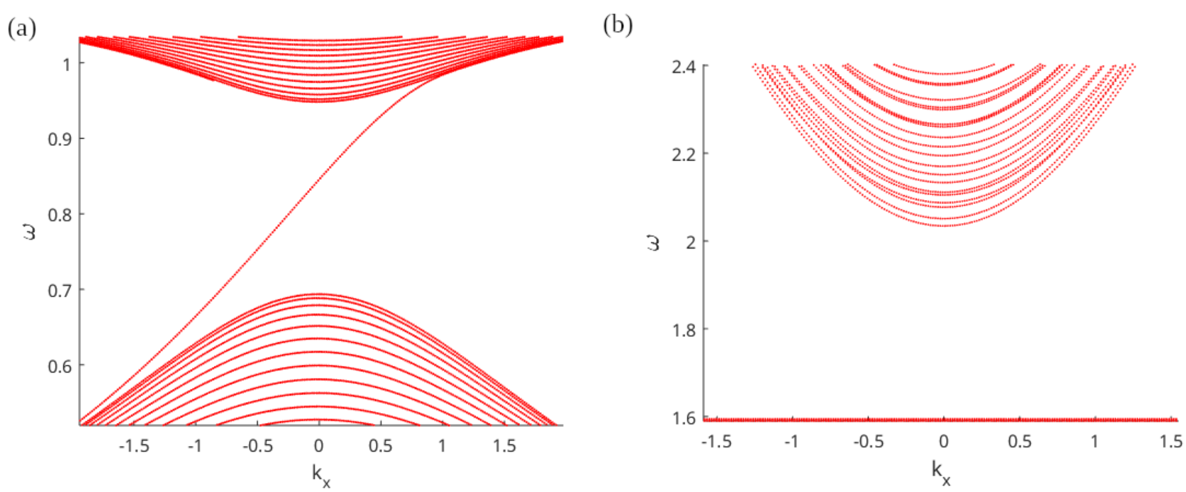



Figure 3: Numerically calculated spectrum with the parameters $(k_z, \Omega) = (2, 1)$ for the transition from phase $\mathrm{I}^+$ to $\mathrm{II}^+$ ($\omega_p$ varying from $0.5\omega_-$ to $1.5\omega_-$ with $\omega_- = 0.8284$ s$^{-1}$). (a) the spectral gap between bands 1 and 2, which illustrates one topologically protected edge state. (b) the spectral gap between bands 2 and 3, which has no edge states as predicted.

## 5.1 Comparison with Numerical Results

Considering parameter values $\Omega(y), \omega_p(y)$ which vary in $y$, we can numerically calculate the eigenvalues of (2) around a boundary between critical values at which band crossings occur by approximating $\partial_y$ by a finite difference matrix and calculating eigenvalues numerically. The numerical scheme is detailed in Appendix C. For transitions with $\pm$ we choose $+$ for concreteness with the understanding that the associated $-$ transition will produce equal and opposite results.

First consider the $\mathrm{I}^+$ to $\mathrm{II}^+$ transition. The numerically calculated spectrum for such a transition is shown in Figure 3 around the band gap between bands 1 and 2 and bands 2 and 3. Figure 3(a) confirms the presence of the TCLW analyzed extensively in [15, 18, 13]. In addition 3(b) shows no topologically protected edge states in the band gap between bands 2 and 3. Both these results agree with our predictions of $(c_1, c_2) = (1, 0)$.

Next consider the $\mathrm{II}^+$ to $\mathrm{III}^+$ transition. From Section 3 we know that $\omega_3(k_\perp = 0) = \omega_p$ and $\omega_2(k_\perp \to \infty) = \sqrt{\omega_p^2 + \Omega^2} \geq \omega_p$. Therefore there is in fact only 1 global band gap possible in this transition, between bands 1 and 2. This band gap is shown in Figure 4, where no edge states are shown by numerical simulations and agree with our prediction of $c_1 = 0$.

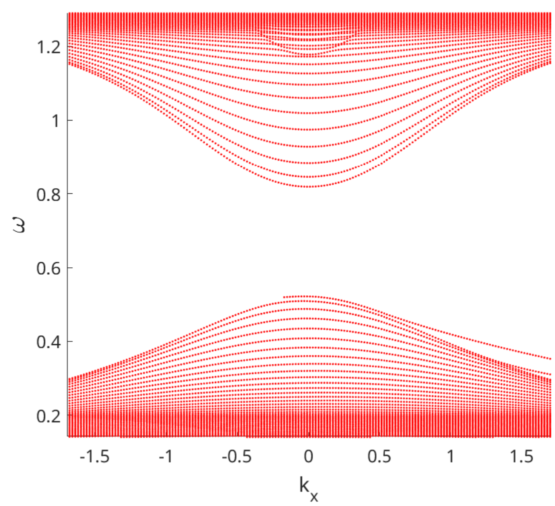



Figure 4: Numerically calculated spectrum with parameters $(k_z, \Omega) = (1, 1)$ for the transition from $\mathrm{II}^+$ to $\mathrm{III}^+$ ($\omega_p$ varying from $0.5\omega_+$ to $1.5\omega_+$ with $\omega_+ = 1.33$ s$^{-1}$).

Next we consider the transition $\mathrm{II}^-$ to $\mathrm{II}^+$. As described in the previous section band 2 does not cross any other band when transitioning from $\mathrm{II}^-$ to $\mathrm{II}^+$ so we expect there to be no topologically protected edge modes for either band gap. This was in fact our motivation for the regularization technique $\Omega(k) \to 0$ as $k \to 0$. Figure 5 confirms numerically our prediction of no edge modes in either band gap.

Finally we consider the band gap between $\omega_2$ and $\omega_4$ for Transverse Magnetic waves as described in Section 4.4.4 when TE and TM modes are decoupled at $k_z = 0$. In this case we can still vary $\Omega$ from negative to positive which is represented by the $\mathrm{IV}^-$ to $\mathrm{IV}^+$ transition. The gap Chern numbers in this case are $(c_1, c_2) = (0, 2)$, predicting two edge modes in the upper gap and none in the lower gap. This prediction is confirmed in Figure 6.

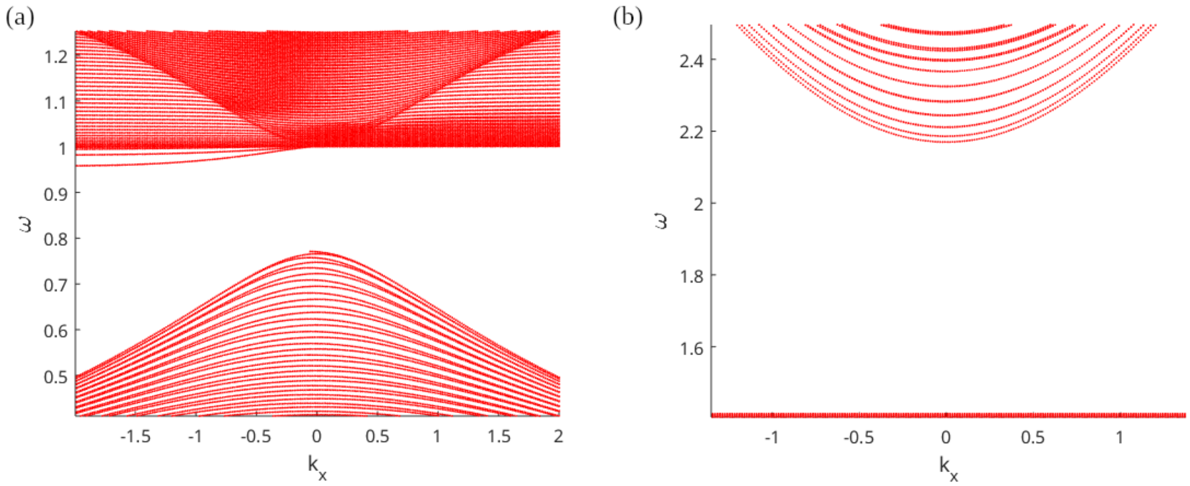



Figure 5: Numerically calculated spectrum with the parameters $(k_z, \omega_p) = (2, 1)$ for the transition from phase II- to II+ with $\Omega$ varying from -0.75 to 0.75 $s^{-1}$. (a) shows the spectral gap between bands 1 and 2 and (b) the gap between bands 2 and 3. No edge states are present as predicted by BDI's.

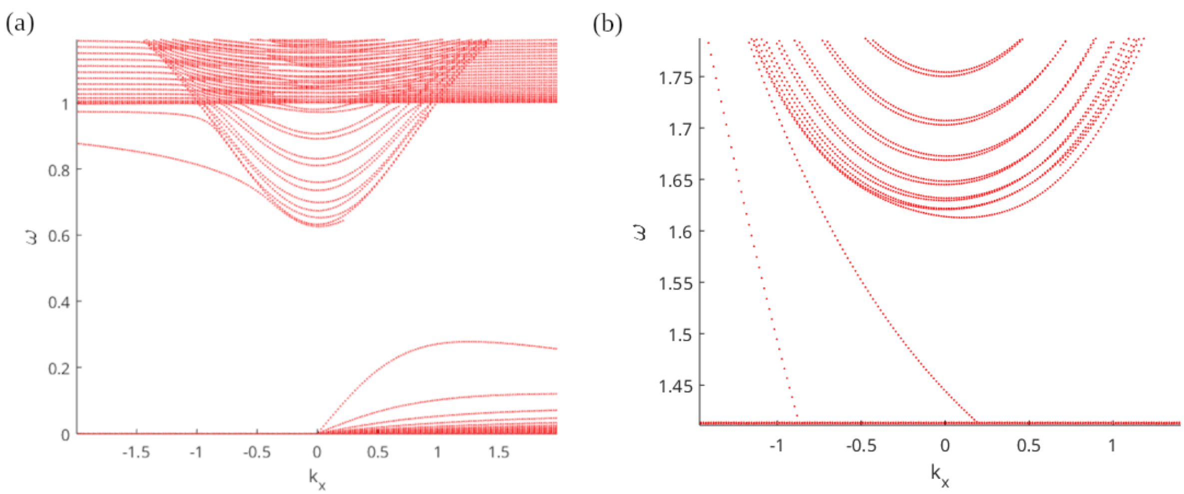



Figure 6: Numerically calculated spectrum with parameters $(k_z, \omega_p) = (0, 1)$ where $\Omega$ varies from -1 to 1 $s^{-1}$. (a) shows the gap between transverse magnetic modes 1 and 2, showing no edge modes. (b) shows the spectral gap between transverse magnetic modes 2 and 3, demonstrating 2 topologically protected edge modes as predicted. Both are consistent with prediction of BDI's.

# 6 Conclusion

Here we have analytically calculated the topological invariants of the cold plasma system. The results we obtain using Chern numbers, which are defined as integers of Berry Curvature over the plane of all perpendicular wave numbers $\mathbf{k}_\perp$, agree with previous results calculated both numerically and analytically for the cold plasma system and closely related photonic systems. It has been a well studied fact since the seminal study [1] that regularization is needed in order to restore integer Chern numbers. The regularization introduced in [1] indeed allowed edge states to be correctly predicted in two important cases [2, 15, 1, 13]. However, our analysis shows that this regularization also predicts edge states incorrectly at some transitions. Namely, both numerical simulations and heuristic arguments based on the absence of band crossings at the II$^-$ to II$^+$ transition show that there are no edge states present at this interface when the previous regularization predicts 2 or more.

In order to remedy this obstruction to the BEC we propose here a regularization that amounts to $\Omega(k) \to 0$ as $k \to 0$, as opposed to the previous regularization which amounts to $\omega_p(k) \to 0$ as $k \to 0$. We find through numerical simulations of the spectrum of interface operators with continuously varying parameters that this regularization leads to correct predictions of edge states in all cases where a global band gap exists.

Of particular note is that continuity is in fact a central assumption in our validation of the BEC in this case. Many recent results [5, 17, 19, 20] have shown that there may be violations of the BEC in the transition IV$^-$ to IV$^+$. Many of these arguments are based on physical limitations of the particular photonic/cold plasma model and non-local effects which are not considered here [17, 19, 20]. However, in [5] it was shown numerically and by Green's function analysis that only one topologically protected edge mode exists in the gapped transition from IV$^-$ to IV$^+$ although 2 were still predicted. While it would seem that our results differ, a key difference in our assumptions is the insistence of continuity of topological phase transitions which are not assumed in [5], e.g. continuity of $\omega_p(y), \Omega(y)$. It was shown in [21] that in a $3 \times 3$ PDE model of shallow water equations which govern equatorial waves the BEC holds only when the parameters of the system are continuous. We expect that a similar result holds here and the effect of boundary regularity on BEC for photonic systems is the subject of future work.

# Appendix A: $\Omega = 0$ Calculations

In this case from (2) with $\Omega = 0$:

$$\begin{aligned} -i\omega_p E &= \omega v \\ i\omega_p v - k \times B &= \omega E \\ k \times E &= \omega B \end{aligned} \tag{17}$$

we can now eliminate $E$ easily:

$$\begin{aligned} E &= i\frac{\omega}{\omega_p} v \\ i\omega_p(1 - \frac{\omega^2}{\omega_p^2})v &= k \times B \\ i\frac{\omega}{\omega_p} k \times v &= \omega B \end{aligned} \tag{18}$$

First consider $\omega = 0$. From (18) we immediately get that $E = 0$, $i\omega_p v = k \times B$, and the third equation becomes trivial. For arbitrary $B \in \mathbb{C}^3$ we can plug these values back into (17), confirming that $-iE = 0 = \omega v$, $k \times E = 0 = \omega B$ and:

$$i\omega_p v - k \times B = i\omega_p(\frac{-i}{\omega_p} k \times B) - k \times B = 0$$

Therefore $\omega = 0$ is an eigenvalue with eigenvector:

$$\omega_0 = 0$$

$$\Psi_0 = (-\frac{i}{\omega_p} k \times B, 0, B)$$

for $B \in \{e_1, e_2, e_3\}$ or any other orthonormal basis of $\mathbb{C}^3$. We have seen that for any parameter values $\Psi = (0, 0, k)$ is a zero eigenvector so it is more natural to choose a basis which includes $k$ and two vectors perpendicular to $k$. However, for arbitrary $k$ there is no natural way to choose these two vectors. Choosing a basis based on $k$ will prove to be essential later.

Next consider the Ansatz $\omega = \omega_p$. Plugging this into (18) gives the eigenvalue/eigenvector pair:

$$\omega_1 = \omega_p$$

$$\Psi_1 = (k, ik, 0)$$

Modes with $\omega = \omega_p$ are often called the plasma or Langmuir oscillations.

Finally, consider the case that $v \perp k$. Then from (18) we have that:

$$\omega B = i\frac{\omega}{\omega_p}(k \times v) \Rightarrow k \times B = \frac{i}{\omega_p} k \times (k \times v) = -\frac{i}{\omega_p} k^2 v$$

and

$$i\omega_p(1 - \frac{\omega^2}{\omega_p^2})v = -\frac{i}{\omega_p} k^2 v \Rightarrow$$

$$(\omega^2 - \omega_p^2)v = k^2 v \Rightarrow$$
$$\omega^2 = k^2 + \omega_p^2$$

Plugging this value into (17) gives the eigenvalue/eigenvector pair:

$$\omega_2 = \sqrt{k^2 + \omega_p^2}$$

$$\Psi_2 = (\omega_p v, i\omega_2 v, ik \times v)$$

We must choose $v \perp k$, but this subspace has dimension 2, so this eigenvector has a multiplicity of 2. To summarize we have found the eigenvalues and eigenvectors:

$$\begin{aligned} \omega_0 &= 0 \\ \Psi_0 &= (-ik \times B, 0, \omega_p B)(3) \\ \omega_1 &= \omega_p \\ \Psi_1 &= (k, ik, 0) \\ \omega_2 &= \sqrt{k^2 + \omega_p^2} \\ \Psi_2 &= (\omega_p v, i\omega_2 v, ik \times v)(2) \end{aligned} \tag{19}$$

Here we have 3 positive eigenvalues, so by the $\pm$ symmetry shown in Section 4.1 we can obtain 3 corresponding negative eigenvalues and their corresponding eigenvectors. Combined with 3 zero eigenvectors we have found all the eigenvectors of the system for $\Omega = 0$. For concreteness assume that $k_y = 0$ so that $\hat{y} \perp \mathbf{k}$. This gets us the eigenvectors:

$$\Psi_1 = (-k(\hat{y} + i\hat{k} \times \hat{y}), 0, \omega_p(\hat{y} + i\hat{k} \times \hat{y}))^T$$
$$\Psi_2 = (k, ik, 0)$$
$$\Psi_3 = (\omega_p \hat{y}, i\omega_3 \hat{y}, ik \times \hat{y})^T$$
$$\Psi_4 = (\omega_p k \times \hat{y}, i\omega_4 k \times \hat{y}, -ik^2 \hat{y})$$

which may be unitarily transformed by $\mathbf{R}$ to obtain eigenvectors for any value of $\mathbf{k}$

Now apply these eigenvectors to calculate Chern numbers. Since $B_0 = 0$ the choice of coordinate basis is arbitrary. Therefore WLOG assume that $k_z \neq 0$. This allows us to denote an orthogonal eigenbasis which is smooth in $k_\perp$ as follows:

$$\omega_0 = 0,\ \omega_1 = \omega_p,\ \omega_2 = \sqrt{k^2 + \omega_p^2}$$

$$\Psi_0^{(1)} = (-ik \times \hat{x}, 0, \omega_p \hat{z})$$
$$\Psi_0^{(2)} = (-ik \times \hat{y}, 0, \omega_p \hat{y})$$
$$\Psi_0^{(3)} = (-ik \times \hat{z}, 0, \omega_p \hat{z})$$
$$\Psi_1 = (k, ik, 0)$$
$$\Psi_2^{(1)} = (k \times \hat{x},\ i\omega_2(k \times \hat{x}),\ ik \times (k \times \hat{x}))$$

$$\Psi_2^{(2)} = (\omega_p k \times \hat{y},\ i\omega_2(\omega_p k \times \hat{y}),\ ik \times (k \times \hat{y}))$$

From (11) and (13) it's easy to see that:

$$F_1 = -2\ \text{Im}\left((\partial_{k_x}\Psi_1)^* \partial_{k_y}\Psi_1\right) = (\hat{x}, -i\hat{x}, 0)\begin{pmatrix}\hat{y}\\ i\hat{y}\\ 0\end{pmatrix} = 0\ \ \forall k$$

$$\Rightarrow C_1 = 0$$

It's also straightforward to calculate:

$\partial_{k_x}(k \times \hat{x}) = 0 \quad \partial_{k_x}(k \times \hat{y}) = \hat{z} \quad \partial_{k_x}(k \times \hat{z}) = -\hat{y}$

$\partial_{k_y}(k \times \hat{x}) = -\hat{z} \quad \partial_{k_y}(k \times \hat{y}) = 0 \quad \partial_{k_y}(k \times \hat{z}) = \hat{x}$

and so

$\partial_{k_x}\Psi_0^{(1)} = (0,0,0) \quad \partial_{k_x}\Psi_0^{(2)} = (-i\hat{z},0,0) \quad \partial_{k_x}\Psi_0^{(3)} = (i\hat{y},0,0)$

$\partial_{k_y}\Psi_0^{(1)} = (i\hat{z},0,0) \quad \partial_{k_y}\Psi_0^{(2)} = (0,0,0) \quad \partial_{k_y}\Psi_0^{(3)} = (-i\hat{x},0,0)$

Therefore again we have:

$$(\partial_{k_x}\Psi_0)^*\Psi_0 = 0 \Rightarrow F_0 = 0\ \ \forall k_\perp \Rightarrow C_0 = 0$$

Finally, for $\Psi_2$ we can calculate:

$$\partial_{k_x}\Psi_2^{(1)} = \Big(0,\quad 0,\quad 0,\quad 0,\quad i\tfrac{2|k|k_x}{\omega_2}k_z,\quad -i\tfrac{2|k|k_x}{\omega_2}k_y,\quad 0,\quad ik_y,\quad ik_z\Big)^T$$

$$\partial_{k_x}\Psi_2^{(1)} = \Big(0,\quad 0,\quad -1,\quad i\tfrac{2|k|k_y}{\omega_2}k_z,\quad -i\tfrac{2|k|k_y}{\omega_2}k_y,\quad -\omega_2,\quad -2ik_y,\quad ik_x,\quad 0\Big)^T$$

Although this time $(\partial_{k_x}\Psi_2)^*\partial_{k_y}\Psi_2 \neq 0$ we can see that each component of $\partial_{k_x}\Psi_2$ and the corresponding component of $\partial_{k_y}\Psi_2$ are either purely imaginary or purely real. Therefore $\text{Im}\left[(\partial_{k_x}\Psi_2)^*\partial_{k_y}\Psi_2\right] = 0 \Rightarrow F_2 = 0\ \ \forall k_\perp$ so again $C_2 = 0$. Rotational symmetry gives the same result for $\Psi_2^{(2)}$ as $\Psi_2^{(1)}$. Therefore if $B_0 = 0$ we have that all bands are topologically trivial.

# Appendix B: $k_z = 0$ Calculations

$$\begin{aligned} -i\Omega\hat{z} \times v - i\omega_p E &= \omega v \\ i\omega_p v - k \times B &= \omega E \\ k \times E &= \omega B \end{aligned} \tag{20}$$

noting in this case $k \perp \hat{z}$. Consider first and zero eigenvectors with $v \parallel \hat{z}$. This gives:

$$-i\Omega\hat{z} \times v - i\omega_p E = 0 \Rightarrow E = 0$$

$$i\omega_p v - k \times B = 0 \Rightarrow k \times B = i\omega_p\hat{z}$$

Assuming that $B \perp k$ we can solve for B:

$$k \times (k \times B) = -k^2 B = i\omega_p k \times \hat{z} \Rightarrow$$

$$B = -\frac{i\omega_p}{k^2} k \times \hat{z}$$

which gives another 0 eigenvector:

$$\Psi_0 = (k^2 \hat{z}, 0, -i\omega_p k \times \hat{z})$$

Now consider a zero eigenvector with $E \parallel k$. Clearly we have $k \times E = 0$ and plugging into the first equation gives:

$$i\omega_p E = i\omega_p k = -i\Omega \hat{z} \times v$$

If we assume that both $E$ and $v$ are perpendicular to $\hat{z}$ then we get:

$$-\Omega \hat{z} \times (\hat{z} \times v) = \Omega v = \omega_p \hat{z} \times k \Rightarrow$$

$$v = \frac{1}{\sigma} \hat{z} \times k$$

Plugging into the second line gives:

$$i\omega_p v = \frac{i\omega_p}{\sigma} \hat{z} \times k = -\frac{i\omega_p}{\sigma} k \times \hat{z} = -k \times B \Rightarrow$$

$$B = \frac{i\omega_p}{\sigma} \hat{z}$$

Therefore the $k_z = 0$ case produces a third zero eigenvector:

$$\Psi_0^{(3)} = (-k \times \hat{z}, \sigma k, i\omega_p \hat{z})$$

Now recall the plasma wave equation, (6). The $k_z = 0$ case corresponds to $\theta = \pi/2$, which yields:

$$\begin{bmatrix} S & iD & 0 \\ -iD & S - n^2 & 0 \\ 0 & 0 & P - n^2 \end{bmatrix} E = 0$$

The first non-trivial solution to this equation is

$$n^2 = P \Rightarrow k^2 = \omega^2 - \omega_p^2 \Rightarrow$$

$$\omega_3 = \sqrt{k^2 + \omega_p^2}$$

Clearly the $E$ associated with this solution is $E = \hat{z}$. Plugging this in to (2) we get:

$$k \times E = \omega_3 B \Rightarrow$$

$$B = \frac{1}{\omega_3} k \times \hat{z}$$

$$i\omega_p v - k \times B = \omega_3 E \Rightarrow$$

$$v = \frac{i\hat{z}}{\omega_p \omega_3}(k^2 - \omega_3^2) = -\frac{i\hat{z}\omega_p}{\omega_3}$$

Summarizing, the eigenvector/eigenvalue pair is:

$$\omega_3 = \sqrt{k^2 + \omega_p^2}$$

$$\Psi_3 = (-i\omega_p \hat{z}, \omega_3 \hat{z}, k \times \hat{z})$$

Due to the fact that $E \perp k$ this wave is often referred to as the Transverse Electric wave [6].

The final eigenvalues solve the equation:

$$S(S - n^2) - D^2 = 0 \Rightarrow$$

$$n^2 = \frac{S^2 - D^2}{S} = \frac{RL}{S}$$

Plugging in the definitions of $n, R, L, S$ gives:

$$\left(\frac{k}{\omega}\right)^2 = \frac{\left(1 - \frac{\omega_p^2}{\omega(\omega+\Omega)}\right)\left(1 - \frac{\omega_p^2}{\omega(\omega-\Omega)}\right)}{\frac{1}{2}\left(2 - \left(\frac{\omega_p^2(\omega+\Omega)+\omega_p^2(\omega-\Omega)}{\omega(\omega^2-\Omega^2)}\right)\right)} \Rightarrow$$

$$k^2 = \frac{\omega^2(\omega^2 - \Omega^2) - 2\omega_p^2\omega^2 + \omega_p^4}{\omega^2 - \Omega^2 - \omega_p^2}$$

Denoting the upper-harmonic frequency $\omega_{uh}^2 = \Omega^2 + \omega_p^2$ we get:

$$k^2 = \frac{\omega^2(\omega^2 - \Omega^2) - 2\omega_p^2\omega^2 + \omega_p^4}{\omega^2 - \omega_{uh}^2}$$

Immediately we see that there as $\omega \to \pm\omega_{uh}$, $k^2 \to \infty$. With some algebraic manipulation we get a quartic equation for the remaining eigenvalues:

$$\omega^4 - \omega^2(k^2 + \Omega^2 + 2\omega_p^2) + (\omega_p^4 + k^2\omega_{uh}^2) = 0$$

Absence of odd-degree terms means we can solve using the quadratic formula. Using the useful substitution $\omega_h^2 = \Omega^2 + 2\omega_p^2$ we get:

$$\omega^2 = \frac{1}{2}\left((k^2 + \omega_h^2) \pm \sqrt{(k^2 + \omega_h^2)^2 - 4(\omega_p^4 + k^2\omega_{uh}^2)}\right)$$

$$= \frac{1}{2}\left((k^2 + \omega_h^2) \pm \sqrt{k^4 + 2(2\omega_p^2 - k^2)\Omega^2 + \Omega^4}\right)$$

This gives us 2 positive and 2 negative bands. We can also see that if $n^2 \neq P$, then we must have $E_z = 0$, so in this case $E$ is polarized in the $x/y$ plane. Since $k \times E = \omega B$, then we see that $B \parallel \hat{z}$. This is why these last two modes are sometimes denoted the Transverse Magnetic waves [6]. From (3) we know that $k \times B = \omega\epsilon E$. Therefore in this case we have:

$$B = -\hat{z}$$

$$E = \epsilon^{-1}\frac{k}{\omega} \times \hat{z}$$

$$v = \frac{i}{\omega_p}(I - \epsilon^{-1})k \times \hat{z}$$

Remembering that:

$$\epsilon = \begin{bmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

with $S$, $D$, and $P$ as defined in Section 3.2 we get:

$$\epsilon^{-1} = \frac{1}{S^2 - D^2}\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & \frac{S^2-D^2}{P} \end{bmatrix}$$

Comparing with the calculations from Section 4.3 we can see that

$$\frac{S}{S^2 - D^2} = \frac{S}{RL} = \frac{\omega^2}{k^2} = \frac{\omega^2(\omega^2 - \omega_{uh}^2)}{\omega^4 - 2\omega_h^2\omega^2 + \omega_p^2}$$

Similarly:

$$\frac{D}{RL} = \frac{\omega_p^2(\omega + \Omega) - \omega_p^2(\omega - \Omega)}{\omega(\omega^2 - \Omega^2)}\frac{\omega^2(\omega^2 - \Omega^2)}{\omega^4 - 2\omega_h^2\omega^2 + \omega_p^2} = \frac{\omega_p^2\Omega\omega}{\omega^4 - 2\omega_h^2\omega^2 + \omega_p^2} = \frac{\omega^2}{k^2}\frac{\omega_p^2\Omega}{\omega(\omega^2 - \omega_{uh}^2)}$$

Plugging these results in we get:

$$\epsilon^{-1} = \frac{\omega^2}{k^2}\begin{bmatrix} 1 & -i\frac{\Omega\omega_p^2}{\omega(\omega^2-\omega_{uh}^2)} & 0 \\ i\frac{\Omega\omega_p^2}{\omega(\omega^2-\omega_{uh}^2)} & 1 & 0 \\ 0 & 0 & \frac{k^2}{\omega^2-\omega_p^2} \end{bmatrix}$$

Define more compact variables $\alpha$ and $\beta$ for the following calculations we get:

$$\epsilon^{-1} = \begin{bmatrix} \alpha & -i\beta & 0 \\ i\beta & \alpha & 0 \\ 0 & 0 & 1/P \end{bmatrix}$$

$$\alpha = \frac{\omega^2}{k^2} = \frac{(\omega^2 - \omega_{uh}^2)\omega^2}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4}$$

$$\beta = \frac{\omega_p^2\Omega\omega}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4} = \frac{\alpha\omega_p^2\Omega}{\omega(\omega^2 - \omega_{uh}^2)}$$

This allows us to explicitly calculate $E$ and $v$ for the transverse magnetic waves:

$$E = \epsilon^{-1}\frac{k}{\omega} \times \hat{z} = \frac{1}{\omega}\begin{bmatrix} \alpha & -i\beta & 0 \\ i\beta & \alpha & 0 \\ 0 & 0 & 1/P \end{bmatrix}\begin{bmatrix} k_y \\ -k_x \\ 0 \end{bmatrix} = \frac{1}{\omega}\begin{bmatrix} i\beta k_x + \alpha k_y \\ i\beta k_y - \alpha k_x \\ 0 \end{bmatrix} = \frac{1}{\omega}(i\beta k + \alpha k \times \hat{z})$$

$$v = \frac{i}{\omega_p}(I - \epsilon^{-1})k \times \hat{z} = \frac{i}{\omega_p}(k \times \hat{z} - \epsilon^{-1}k \times \hat{z}) = \frac{i}{\omega_p}(k \times \hat{z} - \omega E)$$

$$= \frac{i}{\omega_p}(k \times \hat{z} - (\alpha k \times \hat{z} + i\beta k)) = i\frac{(1-\alpha)}{\omega_p}k \times \hat{z} + \frac{\beta}{\omega_p}k$$

So finally we get all eigenvectors and eigenvalues:

$$\omega_0 = 0$$

$$\Psi_0^{(1)} = (0, 0, \hat{k}_\perp) \;\; \Psi_0^{(2)} = (k_\perp^2 \hat{z}, 0, -i\omega_p k_\perp \times \hat{z}) \;\; \Psi_0^{(3)} = (k_\perp \times \hat{z}, -\sigma k_\perp, i\omega_p \hat{z})$$

$$\omega_2^2 = \frac{1}{2}\left((k^2 + \omega_h^2) - \sqrt{(k^2 + \omega_h^2)^2 - 4(\omega_p^4 + k^2\omega_{uh}^2)}\right)$$

$$\Psi_2 = \left(i\frac{(1-\alpha_2)}{\omega_p}k_\perp \times \hat{z} + \frac{\beta_2}{\omega_p}k_\perp, \frac{1}{\omega_2}(i\beta_2 k_\perp + \alpha_2 k_\perp \times \hat{z}), -\hat{z}\right)$$

$$\omega_3 = \sqrt{k_\perp^2 + \omega_p^2}$$

$$\Psi_3 = (-i\omega_p \hat{z}, \omega_3 \hat{z}, k_\perp \times \hat{z})$$

$$\omega_4 = \frac{1}{2}\left((k^2 + \omega_h^2) + \sqrt{(k^2 + \omega_h^2)^2 - 4(\omega_p^4 + k^2\omega_{uh}^2)}\right)$$

$$\Psi_4 = \left(i\frac{(1-\alpha_4)}{\omega_p}k_\perp \times \hat{z} + \frac{\beta_4}{\omega_p}k_\perp, \frac{1}{\omega_4}(i\beta_4 k_\perp + \alpha_4 k_\perp \times \hat{z}), -\hat{z}\right)$$

$$\alpha_j = \frac{\omega_j^2}{k^2} = \frac{(\omega_j^2 - \omega_{uh}^2)\omega_j^2}{\omega_j^4 - \omega_j^2\omega_h^2 + \omega_p^4}$$

$$\beta_j = \frac{\omega_p^2 \Omega \omega}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4}$$

$$\omega_0 = 0$$

$$\Psi_0^{(1)} = (0, 0, \hat{k}_\perp) \;\; \Psi_0^{(2)} = (k_\perp^2 \hat{z}, 0, -i\omega_p k_\perp \times \hat{z}) \;\; \Psi_0^{(3)} = (k_\perp \times \hat{z}, -\sigma k_\perp, i\omega_p \hat{z})$$

$$\omega_3 = \sqrt{k_\perp^2 + \omega_p^2}$$

$$\Psi_3 = (-i\omega_p \hat{z}, \omega_3 \hat{z}, k_\perp \times \hat{z})$$

$$\omega_{2,4} = \frac{1}{2}\left((k^2 + \omega_h^2) \pm \sqrt{(k^2 + \omega_h^2)^2 - 4(\omega_p^4 + k^2\omega_{uh}^2)}\right)$$

$$\Psi_{2,4} = \left(i\frac{(1-\alpha)}{\omega_p}k_\perp \times \hat{z} + \frac{\beta}{\omega_p}k_\perp, \frac{1}{\omega}(i\beta k_\perp + \alpha k_\perp \times \hat{z}), -\hat{z}\right)$$

$$\alpha = \frac{\omega^2}{k^2} = \frac{(\omega^2 - \omega_{uh}^2)\omega^2}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4}$$

$$\beta = \frac{\omega_p^2 \Omega \omega}{\omega^4 - \omega^2\omega_h^2 + \omega_p^4}$$

Using (13) we can calculate as in the last section:

$$\partial_{k_x}\Psi_0^{(1)} = (0,0,\hat{x}) \quad \partial_{k_x}\Psi_0^{(2)} = (-\hat{y},\sigma\hat{x},0) \quad \partial_{k_x}\Psi_0^{(3)} = (2k_x\hat{z},0,-i\hat{y})$$

$$\partial_{k_y}\Psi_0^{(1)} = (0,0,\hat{y}) \quad \partial_{k_y}\Psi_0^{(2)} = (\hat{x},\sigma\hat{y},0) \quad \partial_{k_y}\Psi_0^{(3)} = (2k_y\hat{z},0,i\hat{x})$$

$$\Rightarrow (\partial_{k_x}\Psi_0^*)\partial_{k_y}\Psi_0 = 0$$

for all values of $k$, Therefore $C_0 = 0$. For $\Psi_3$, the transverse electric mode, we have:

$$\partial_{k_x}\Psi_3 = (0, \frac{2|k|k_x}{\omega_1}\hat{z}, -\hat{y})$$

$$\partial_{k_y}\Psi_3 = (0, \frac{2|k|k_y}{\omega_1}\hat{z}, \hat{x})$$

$$F_3(k) = -2\mathrm{Im}\left(\frac{4k^2k_xk_y}{\omega_3^2}\right) = 0$$

$$\Rightarrow C_3 = 0$$

Finally, for the transverse magnetic modes $\Psi_2$, $\Psi_4$ we will use (14). First notice that our calculations in Section 3.4 do not change if $k_z = 0$. Therefore we get:

$$\lim_{k_\perp\to\infty}\frac{\Psi_2^* ger(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} = \frac{\sigma}{\sqrt{\sigma^2+1}}$$

$$\lim_{k_\perp\to\infty}\frac{\Psi_4^* ger(\mathbf{Z}\times\Psi_4)}{|\Psi_4|^2} = 0$$

As $k \to 0$ we get that:

$$\omega_2^2 \to \frac{\omega_h^2}{2} - \frac{1}{2}\sqrt{4\omega_p^2\Omega^2+\Omega^4} = \omega_{0-}^2$$

and:

$$\omega_4^2 \to \frac{\omega_h^2}{2} + \frac{1}{2}\sqrt{4\omega_p^2\Omega^2+\Omega^4} = \omega_{0+}^2$$

It will also be useful to define $y = \frac{1}{2}\sqrt{4\omega_p^2\Omega^2+\Omega^4}$. Looking closely we can see that $\omega_{0\pm}$ are actually the zeros of $\omega^4 - \omega^2\omega_h^2 + \omega_p^4$ so $\alpha, \beta \to \infty$ as $k \to 0$. Therefore we can calculate:

$$\frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} = i\frac{v^*(\hat{z}\times v) + E^*(\hat{z}\times E) + B^*(\hat{z}\times B)}{|v|^2+|E|^2+|B|^2}$$

$$= i\frac{-i\frac{2\beta(\alpha-1)k^2}{\omega_p^2} - i\frac{2\alpha\beta k^2}{\omega^2}}{\frac{k^2}{\omega_p^2}((1-\alpha)^2+\beta^2) + \frac{1}{\alpha}(\alpha^2+\beta^2) + 1} = \frac{-2(\omega_p^2+\alpha k^2-k^2)}{\frac{\omega_p^2}{\beta} + \frac{k^2}{\beta}((1-\alpha)^2+\beta^2) + \frac{\omega_p^2}{\alpha\beta}(\alpha^2+\beta^2)}$$

$$= \frac{2(\omega_p^2+\omega_{0-}^2)}{\frac{\omega_p^2}{\beta} + \frac{\omega_{0-}^2\alpha}{\beta} - 2\frac{\omega_{0-}^2}{\beta} + \frac{k^2}{\beta} + k^2\beta + \omega_p^2(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})} = \frac{2(\omega_p^2+\omega_{0-}^2)}{(\omega_p^2+\omega_{0-}^2)(\frac{\alpha}{\beta}+\frac{\beta}{\alpha})} = \frac{2}{\frac{\alpha}{\beta}+\frac{\beta}{\alpha}}$$

Here we have made use of the fact that $\alpha k^2 = \omega^2$ and $\beta k^2 = \beta\omega^2/\alpha$. Now we must find $\lim_{k\to 0} \alpha/\beta$:

$$\lim_{k\to 0} \frac{\alpha}{\beta} = \frac{\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)}{\omega_p^2 \Omega}$$

We will first show that $\omega_{0^-}^2(\omega_{0^-}^2 - \omega_{uh}^2)^2 = \omega_p^4\Omega^2$. Plugging in definitions we get:

$$\omega_{0^-}^2 - \omega_{uh}^2 = \omega_p^2 + \frac{\Omega^2}{2} - y - \omega_p^2 - \Omega^2 = -\frac{\Omega^2}{2} - y$$

$$\Rightarrow (\omega_{0^-}^2 - \omega_{uh}^2)^2 = \frac{\Omega^4}{4} + y^2 + \Omega^2 y = \frac{\Omega^4}{4} + \omega_p^2\Omega^2 + \frac{\Omega^4}{4} + \Omega^2 y = \Omega^2(\frac{\omega_h^2}{2} + y)$$

Therefore we get:

$$\omega_{0^-}^2(\omega_{0^-}^2 - \omega_{uh}^2)^2 = \Omega^2(\frac{\omega_h^2}{2}+y)(\frac{\omega_h^2}{2}-y) = \Omega^2(\frac{\omega_h^4}{4}-y^2) = \Omega^2(\frac{4\omega_p^4 + 4\Omega^2\omega_p^2 + \Omega^4}{4} - (\omega_p^2\Omega^2 + \frac{\Omega^4}{4})) = \omega_p^4\Omega^2$$

$$\Rightarrow \omega_p^2\Omega = \pm\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)$$

From Appendix B we know that $\omega_{0^-}^2 \le \omega^2 \le \omega_{uh}^2$. Therefore, assuming for now that $\Omega > 0$ we have that:

$$\omega_p^2\Omega = -\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)$$

since $\omega_p^2\Omega > 0$ and $\omega_{0^-}^2 - \omega_{uh}^2 < 0$. Therefore for the lower TM mode we get:

$$\lim_{k\to 0} \frac{\alpha}{\beta} = \lim_{k\to 0} \frac{\beta}{\alpha} = \frac{\omega_{0^-}(\omega_{0^-}^2 - \omega_{uh}^2)}{\omega_p^2\Omega} = -1$$

This gives:

$$\lim_{k\to 0} \frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} = \frac{2}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} = -1$$

Repeating the above analysis with $\omega \to \omega_{0^+}$ gives identical results except that $\omega_{0^+} > \omega_{uh}$ so:

$$\omega_p^2\Omega = \omega_{0^+}(\omega_{0^+}^2 - \omega_{uh}^2)$$

Therefore:

$$\lim_{k\to 0} \frac{\alpha}{\beta} = \lim_{k\to 0} \frac{\beta}{\alpha} = \frac{\omega_{0^+}(\omega_{0^+}^2 - \omega_{uh}^2)}{\omega_p^2\Omega} = 1$$

and we get:

$$\lim_{k\to 0} \frac{i\Psi_3^*(\mathbf{Z}\times\Psi_3)}{|\Psi_3|^2} = \frac{2}{\frac{\alpha}{\beta} + \frac{\beta}{\alpha}} = 1$$

Finally, plugging these results into (16) gives:

$$C_4 = \lim_{k\to\infty} \frac{i\Psi_4^*(\mathbf{Z}\times\Psi_4)}{|\Psi_4|^2} - \lim_{k\to 0} \frac{i\Psi_4^*(\mathbf{Z}\times\Psi_4)}{|\Psi_4|^2} = -1$$

and for the lower harmonic:

$$C_2 = \lim_{k\to\infty} \frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} - \lim_{k\to 0} \frac{i\Psi_2^*(\mathbf{Z}\times\Psi_2)}{|\Psi_2|^2} = \frac{\sigma}{\sqrt{1+\sigma^2}} + 1$$

# Appendix C: Numerical simulations

We follow a similar finite difference scheme as [15][21] to numerically calculate the spectrum of the interface Hamiltonian $H_I$. We choose to vary the parameters $\omega_p, \Omega$ in the $y$-direction so that the Hamiltonian is invariant with respect to translations in $x$. Taking the Fourier transform of (1) in $t, x, z$ gives:

$$\omega\psi(y) = \begin{pmatrix} i\Omega(y)\hat{e}_z\times & -i\omega_p(y) & 0 \\ i\omega_p(y) & 0 & i\partial_y\hat{e}_y \times -(k_x, 0, k_z)^t\times \\ 0 & -i\partial_y\hat{e}_y \times +(k_x, 0, k_z)^t\times & 0 \end{pmatrix}\psi(y)$$

which is an ODE in $y$ for each $k_x$. Discretizing $y$ into $N$ points on an interval $[-L, L]$ allows us to approximate the equation as:

$$\omega\psi(y_i) = H_i^-\psi(y_{i-1}) + H_i\psi(y_i) + H_i^+\psi(y_{i+1}) \tag{21}$$

$$H_i = \begin{pmatrix} i\Omega(y_i)\hat{e}_z\times & -i\omega_p(y_i) & 0 \\ i\omega_p(y_i) & 0 & \frac{i}{\Delta y}\hat{e}_y \times -\frac{1}{2}(k_x, 0, k_z)^t\times \\ 0 & -\frac{i}{\Delta y}\hat{e}_y \times +\frac{1}{2}(k_x, 0, k_z)^t\times & 0 \end{pmatrix}$$

$$H_i^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -D \\ 0 & 0 & 0 \end{pmatrix} \qquad H_i^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D & 0 \end{pmatrix} \quad D = \frac{i}{\Delta y}\hat{e}_y \times +\frac{1}{2}(k_x, 0, k_z)^t\times$$

where $\Delta y = 2L/N$. We adopt the particular combination of forward and backward differences and averaging $E, B$ in the second and third row from [15] to ensure the discrete problem obeys the same particle-hole symmetry as the continuous one. This convention amounts to discretizing $B$ on half-integer grid points. Calculating the eigenvalues of $H_I$ can then be done by diagonalizing a $9N \times 9N$ matrix. In order to ensure the matrix is Hermitian periodic boundary conditions are enforced ($\psi(-L) = \psi(L)$, $\omega_p(-L) = \omega_p(L)$, $\Omega(-L) = \Omega(L)$). Assuming our parameters cross from one topological phase into another, having periodic parameters $\Omega, \omega_p$ introduces a second topological transition in the opposite direction, which will necessarily produce edge states concentrated around this second edge (in cases where edge states exist). For clarity, we adopt the strategy used in [21] and eliminate eigenvalues whose associated eigenvector has more than half its weight within $0.1L$ of the spurious edge.

One disadvantage of this method is the appearance of continuous branches of spectrum which are edge modes concentrated at $y = 0$ for $k_x < 0$ ($k_x > 0$) and concentrated at $y = L$ for $k_x > 0$ ($k_x < 0$). Figure 8 illustrates one such branch. Since we have two topological phase transitions in opposite directions, we expect that each uni-directional edge state localized at $y = 0$ for $k_{x0} < 0$ will be mirrored by a uni-directional edge state localized at $y = L$ for $k_{xL} = -k_{x0}$. For bands which cross a spectral gap (topologically protected edge states) this produces a mirrored band localized at $y = L$ which produces an identical spectral flow in the opposite direction (see [15, 13]). However for edge states whose energy does not overlap with the spectral gap, bands may appear to switch from concentrated at $y = 0$ for $k_x < 0$ ($k_x > 0$) to concentrated at $y = L$ for $k_x > 0$ ($k_x < 0$) as we see in Figure 9. The appearance of these bands is therefore due to the periodic boundary conditions we

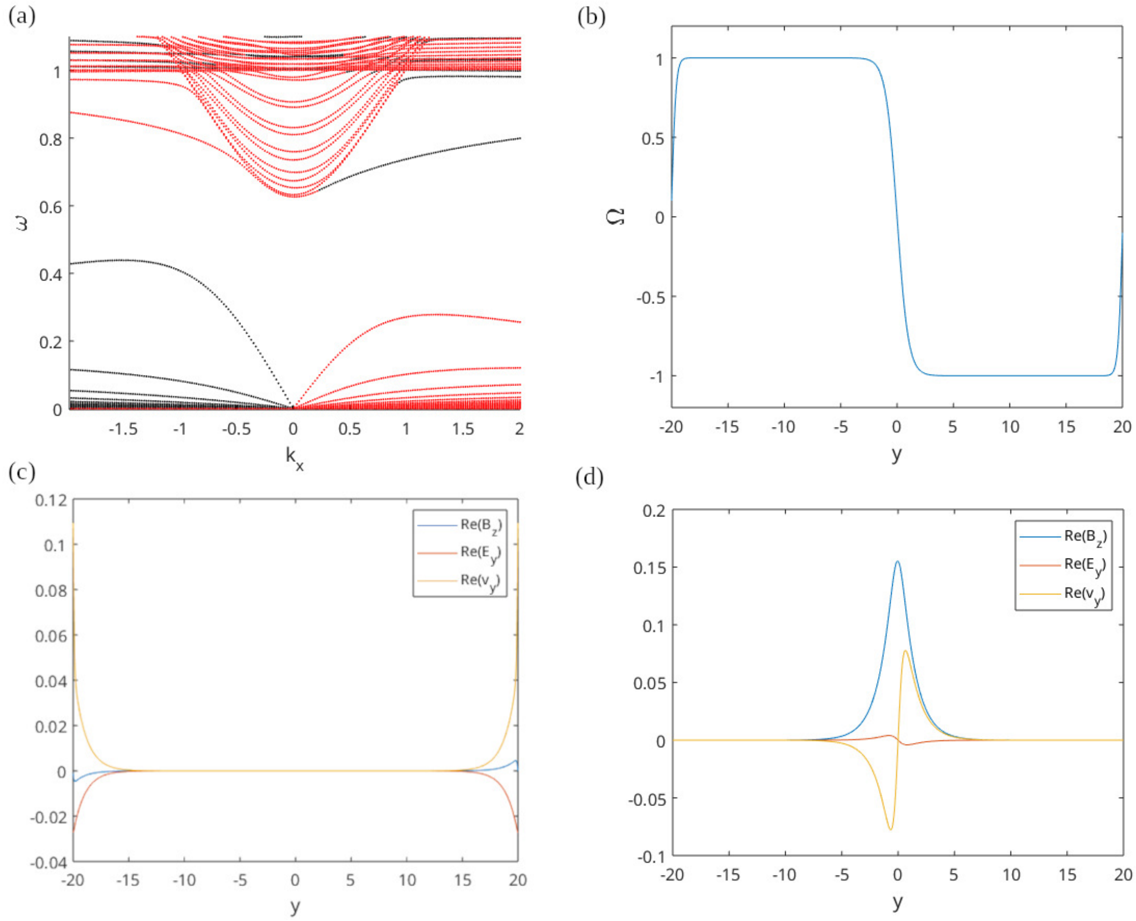



Figure 7: Illustration of the elimination of spurious edge modes for the spectrum calculated in Figure 6(a). (a) shows the spectrum with spurious modes shown in black and bulk and legitimate edge modes in red. (b) is the profile of $\Omega$, which consists of a periodic combination of logistic functions. (c) and (d) plot components of the numerically calculated eigenvectors plotted as a function of $y$. (c) is a mode concentrated around the spurious edge, specifically the eigenvector associated to $(k_x, \omega) = (-0.5, 0.2636)$, while (d) is a legitimate edge mode at $((k_x, \omega) = (0.5, 0.1919))$. Modes of the former type are eliminated for clarity as discussed above.

impose and we expect that such bands would not appear when considering more realistic boundary conditions. Inclusion of the spurious modes shows as in Figures 8 and 9 shows that these branches indeed only exist at energies outside the spectral gap and therefore do not contribute to the number of edge modes which cross the bad gap.

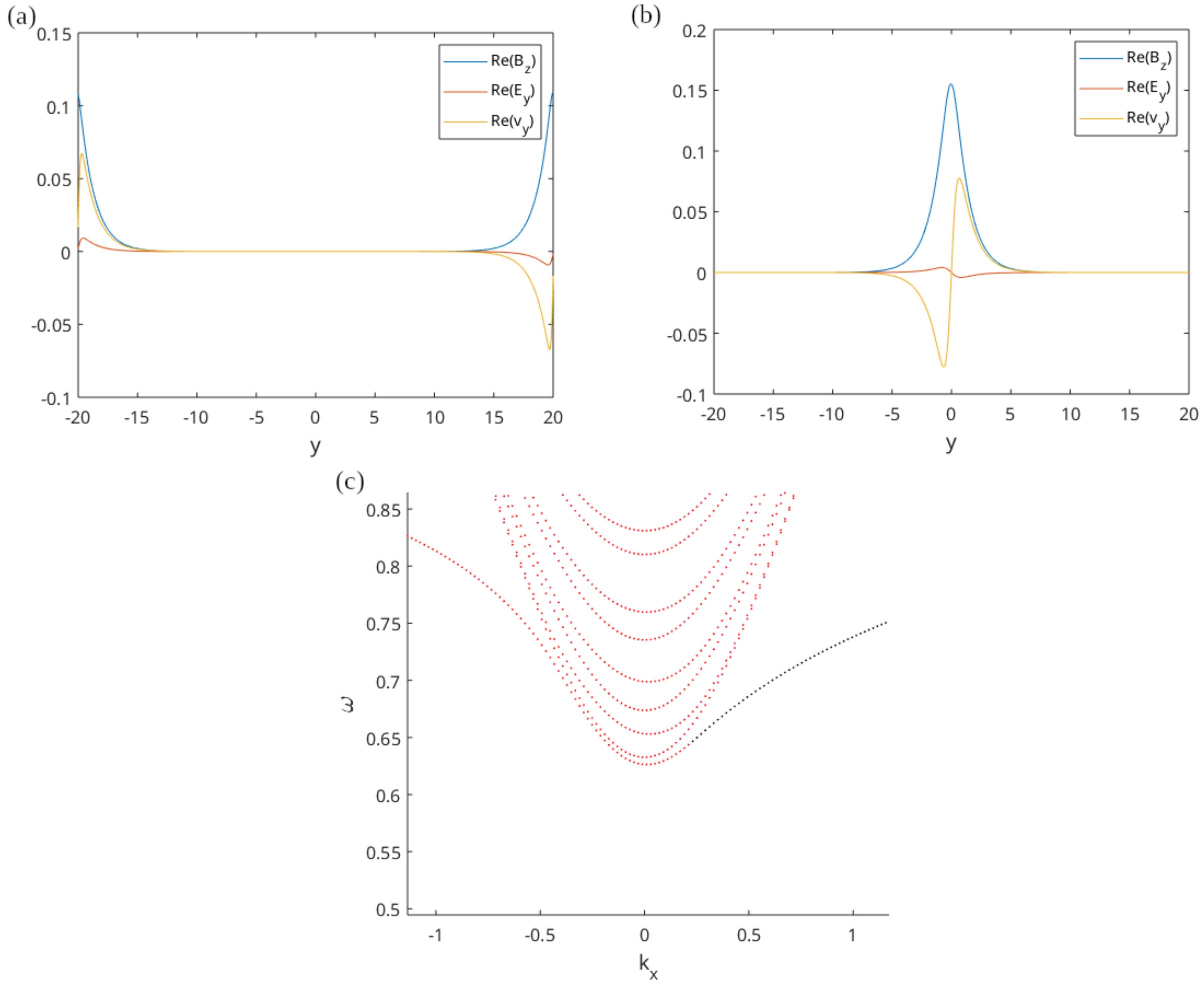



Figure 8: Illustration of branches described in the last paragraph of Appendix E which exhibit both spurious and edge modes. (c) is an inset of the spectrum showed in Figure 7 showing the bottom-most branch transitioning continuously from a spurious mode to a valid edge mode. (a) shows the eigenvector associated to this branch at $k_x = 1$ and (b) the eigenvectors associated to this branch at $k_x = -1$.

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